Sensitivity Analysis of Quasi-Monte Carlo methods for the Heston Model

Emanouil Atanassov[1] Sergei Kucherenko[2] Dimitar Dimitrov[1]

[1]IICT-BAS, Bulgaria

[2]Imperial College London, UK

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The Heston Stochastic Volatility Model

Definition

Definition of the Heston model.

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_s$$

$$dv_t = \kappa \left(\theta v_t\right) dt + \sigma_v \sqrt{v_t} dW_v,$$

dW_s and dW_v are Brownian motions that are correlated with coefficient ρ. Parameters of the model are $\kappa, \theta, \sigma, S_0, V_0, \rho, r$.

Using QMC for option pricing

- Quasi-Monte Carlo sequences have been used extensively for pricing all kinds of financial options.

Using QMC for option pricing

- Quasi-Monte Carlo sequences have been used extensively for pricing all kinds of financial options.
- Despite the high dimensionality of the underlying integrals, they achieve good practical results.
- Various families of low-discrepancy sequences are in wide use for pricing options.
- We concentrate on the Sobol' and Halton sequences.

Using QMC for option pricing

- Scrambling of low-discrepancy sequences has been shown to provide both theoretical and practical advantages.
- Some types of scrambling can be applied with small, even negligible computational cost.
- Owen-type scrambling can be more effective, but uses more resources.
- The construction of the modified Halton sequences, as proposed by Atanassov, can also be considered as a type of scrambling. The leading constant for their star-discrepancy is less than

$$\frac{1}{s!}\left(\sum_{i=1}^{s}\log p_i\right)\prod_{i=1}^{s}\frac{p_i(1+\log p_i)}{(p_i-1)\log p_i}.$$

Lord's full truncation scheme

Lord investigated a variety of Euler-type schemes and how to deal with the possibility of negative volatility.

- $V_t = V_s + k\Delta t\,(\theta - V_s^+) + \sigma\sqrt{V_s^+}Z_v\sqrt{\Delta t}$
- $S_t = S_s(1 + r\Delta t + \sqrt{V_t^+}Z_s\sqrt{\Delta t})$
- More precise:

$$\log S_t = \log S_s + \left(r - \frac{1}{2}V_s^+\right)\Delta t + \sqrt{V_s^+}Z_s\sqrt{\Delta t}.$$

Milstein and Kahl-Jaeckel schemes

Schemes with higher order of approximation have been proposed.

- Milstein:

$$V_t = V_s + k\Delta t\,(\theta - V_s) + \sigma\sqrt{V_s}Z_v\sqrt{\Delta t} + \frac{1}{4}\sigma^2(Z^2-1)\Delta t$$

.

- Kahl-Jaeckel:

$$V_t = \frac{V_s + k\Delta t\theta + \sigma\sqrt{V_s}Z_v\sqrt{\Delta t} + \frac{1}{4}\sigma^2(Z^2-1)\Delta t}{1+\kappa\Delta t}.$$

Andersen scheme

- The Andersen scheme is attempt to approximate the scheme of Broadie and Kaya, which is computationally very intensive.
- The sampling procedure requires for each step the sampling of 1 normally distributed number and 1 number that is sampled differently depending on certain condition.
- Thus one can expect that this scheme will not be suitable for use of QMC. However, this is not the case in practice.

Sobol' sensitivity indices and sensitivity analysis

- Based on an ANOVA decomposition of a function

$$f(x) = f_0 + \sum i = 1^d f_i(x_i) + \cdots \sum_{i=1}^{d} \sum_{j=i+1}^{d} f_{ij}(x_i, x_j) \ldots$$

  Sobol' defined coefficients

$$S_{i_1,\ldots,i_k} = D_{i_1,\ldots,i_k}/D,$$

  which measure the sensitivity of the function to certain subsets of variables.
- These coefficients can be estimated numerically via Monte Carlo or Quasi-Monte Carlo methods.
- In this work we used the so-called Oracle formula and QMC sequences (Sobol' and Halton).

Sobol' sensitivity indices for numerical schemes for Heston

- Each scheme can be considered as integration scheme for a suitable function from the $2d$-dimensional unit cube.
- Different types of options lead to different sensitivity indices.
- Some exotic options are more problematic for the QMC methods.
- We computed Sobol sensitivity indices for 1 dimension, 2 consecutive dimensions and the total sensitivity indices.

Numerical results - accuracy of option pricing

- European option (has analytical solution), parameters $r = 3.19\%, \kappa = 6.21, \theta = 0.019, \sigma_v = 0.61, \rho = 0.7, S_0 = 100, V_0 = 0.010201$

Nstep	Milstein	Kahl-Jaeckel	Andersen
250	0.054	0.047	0.038
500	0.031	0.024	0.024
750	0.015	0.013	0.011
1000	0.010	0.011	0.007

Table: Error from computation with varying number of steps

Numerical results - accuracy of option pricing

- European option (has analytical solution), parameters $r = 3.19\%, \kappa = 6.21, \theta = 0.019, \sigma_v = 0.61, \rho = 0.7, S_0 = 100, V_0 = 0.010201$

N	MC	Sobol	Halton
256	0.60	0.23	0.13
512	0.23	0.18	0.08
1024	0.13	0.09	0.04
2048	0.08	0.01	0.02

Table: Error from computation with 12 steps

N	MC	Sobol	Halton
256	0.55	0.23	0.10
512	0.26	0.16	0.07
1024	0.17	0.11	0.03
2048	0.05	0.02	0.01

Table: Error from computation with 52 steps

Numerical results - accuracy of option pricing

- The so-called "Greeks" can also be computed with QMC via numerical differenciation. The advantages of QMC are not so pronounced in that case.
- Asian option, Lord's full truncation, parameters

$$r = 0.\%, \kappa = 1.0606, \theta = 0.0733, \sigma_v = 0.3918, \rho = 0.3456, S_0 = 100., V_0$$

N	MC	Sob.	Hal.	MC	Sob.	Hal.
256	0.61	0.12	0.40	1.02	.66	0.85
512	0.53	0.06	0.20	.75	0.44	0.59
1024	0.31	0.07	0.12	0.45	0.50	0.44

Table: Error from computation with 32 steps (effective dimension 64), price and delta

Numerical results - accuracy of option pricing

- Asian option, Andersen scheme, parameters

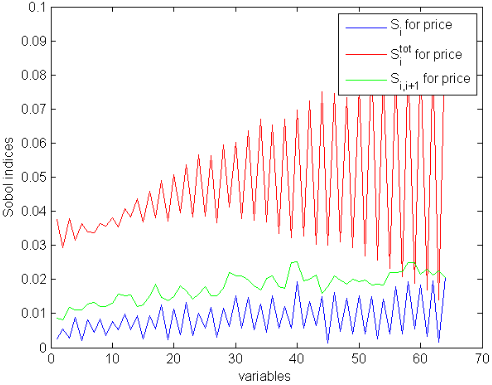
$$r = 0.\%, \kappa = 1.0606, \theta = 0.0733, \sigma_v = 0.3918, \rho = 0.3456,$$

$$S_0 = 100., V_0 = 0.0222, K = 100.$$

N	MC	Sob.	Hal.	MC	Sob.	Hal.
256	0.72	0.14	0.44	1.31	.81	1.01
512	0.65	0.09	0.21	.98	0.63	0.76
1024	0.39	0.07	0.13	0.59	0.54	0.52

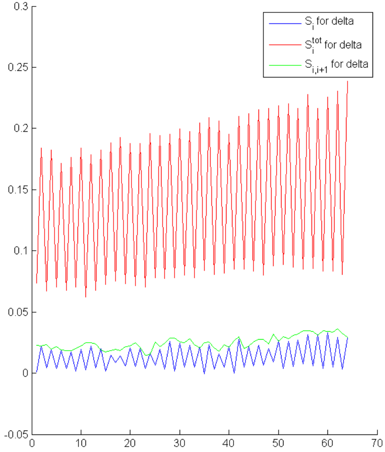
Table: Error from computation with 32 steps (effective dimension 64), price and delta

Numerical results - Sobol' sensitivity indices



Sobol' sensitivity indices for the price.

Numerical results - Sobol' sensitivity indices



Sobol' sensitivity indices for the delta.

Summary

- The various numerical schemes used for option pricing via the Heston model can benefit from use of low-discrepancy sequences.
- The Sobol' sequence with Owen scrambling and the modified Halton sequences proved to be effective.
- The Sobol' sensitivity coefficients reveal the strength of interactions between variables and consequently, coordinates of the low-discrepancy sequences.
- It remains to be seen how the Brownian bridge construction can work in situations not all variables have normal distribution.