Stochastic integration in quasi-Banach spaces: Besov regularity of the stochastic heat equation

Sonja Cox (University of Amsterdam)

Joint work with Petru Cioica (University of Otago, NZ), and Mark Veraar (TU Delft, NL)

MCQMC 2018, Rennes

UNIVERSITY OF AMSTERDAM

Question: what spatial Besov regularity does the solution to the stochastic heat equation posess?

Question: what spatial Besov regularity does the solution to the stochastic heat equation posess?

I.e., let

- $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ a filtered probability space,
- $d \in \mathbb{N}$, $p, q \in (0, \infty)$, $\tau \in \mathbb{R}$,
- $W_n \colon [0, \infty) \times \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, independent standard $\mathbb{F}$-Brownian motions,
- $g_n \colon [0, \infty) \times \Omega \to B^{\tau}_{p,q}(\mathbb{R}^d)$, $n \in \mathbb{N}$, $\mathbb{F}$-progressively measurable, and
- $u_0 \in B^{\tau+\frac{1}{2}}_{p,q}(\mathbb{R}^d)$.

Consider

$$\begin{cases} \mathrm{d}u = \Delta u \, \mathrm{d}t + \sum_{n \geq 1} g_n \, \mathrm{d}W_n(t), t \in [0, \infty), \\ u(0) = u_0. \end{cases}$$

Question: for what $\sigma \in \mathbb{R}$ and in what sense does a solution $u \colon [0, T] \times \mathbb{R} \to B^{\sigma}_{p,q}(\mathbb{R}^d)$ to this equation exist?

Why care about Besov regularity?

The 'σ' in $B^{\sigma}_{p,q}(\mathbb{R}^d)$ indicates the smoothness of elements of this space.

[1] See DeVore (1998), Binev et al. (2002), Gaspoz and Morin (2014)

Why care about Besov regularity?

The 'σ' in $B^{\sigma}_{p,q}(\mathbb{R}^d)$ indicates the smoothness of elements of this space.

More specifically, it determines the efficiency of adaptive discretization algorithms[1] for a different range of parameters as the efficiency of non-adaptive discretization algorithms.

[1] See DeVore (1998), Binev et al. (2002), Gaspoz and Morin (2014)

Optimal convergence rate in $\|\cdot\|_{L^q(\mathbb{R}^d)}$ for an adaptive wavelet/finite element approximation is determined by greatest α s.t. the target function lies in $B^{\alpha}_{p,p}(\mathbb{R}^d)$ $(= W^{\alpha,p}(\mathbb{R}^d))$[2], where $\frac{1}{p} = \frac{\alpha}{d} + \frac{1}{q}$.

Optimal convergence rate in $\|\cdot\|_{L^q(\mathbb{R}^d)}$ for non-adaptive wavelet/finite element approximation is determined by greatest α s.t. the target function lies in $W^{\alpha,q}(\mathbb{R}^d)$.

[2] if $\alpha \notin \mathbb{N}$

Optimal convergence rate in $\|\cdot\|_{L^q(\mathbb{R}^d)}$ for an adaptive wavelet/finite element approximation is determined by greatest α s.t. the target function lies in $B^\alpha_{p,p}(\mathbb{R}^d)$ $(= W^{\alpha,p}(\mathbb{R}^d))$[2], where $\frac{1}{p} = \frac{\alpha}{d} + \frac{1}{q}$.

Optimal convergence rate in $\|\cdot\|_{L^q(\mathbb{R}^d)}$ for non-adaptive wavelet/finite element approximation is determined by greatest α s.t. the target function lies in $W^{\alpha,q}(\mathbb{R}^d)$.

Conclusion: adaptive methods may outperform non-adaptive methods.

[2] if $\alpha \notin \mathbb{N}$

Optimal convergence rate in $\|\cdot\|_{L^q(\mathbb{R}^d)}$ for an adaptive wavelet/finite element approximation is determined by greatest α s.t. the target function lies in $B^{\alpha}_{p,p}(\mathbb{R}^d)$ $(= W^{\alpha,p}(\mathbb{R}^d))^2$, where $\frac{1}{p} = \frac{\alpha}{d} + \frac{1}{q}$.

Optimal convergence rate in $\|\cdot\|_{L^q(\mathbb{R}^d)}$ for non-adaptive wavelet/finite element approximation is determined by greatest α s.t. the target function lies in $W^{\alpha,q}(\mathbb{R}^d)$.

Conclusion: adaptive methods may outperform non-adaptive methods.

Difficulty: if $\alpha > \frac{d(q-1)}{q}$ then $B^{\sigma}_{p,p}$, $p \in (0,1)$, is a **quasi-Banach space**, *not* a Banach space.
More precisely, for $p \in (0,1)$, $\sigma \in \mathbb{R}$, $x, y \in B^{\sigma}_{p,p}$ one has

$$\|x+y\|^p_{B^{\sigma}_{p,p}} \leq \|x\|^p_{B^{\sigma}_{p,p}} + \|y\|^p_{B^{\sigma}_{p,p}},$$

i.e., $\|\cdot\|_{B^{\sigma}_{p,p}}$ is an *p-norm*.

[2] if $\alpha \notin \mathbb{N}$

Challenge: develop stochastic calculus in quasi-Banach space setting. More about this later. First we answer the well-posedness question that was solved using this stochastic calculus.

Challenge: develop stochastic calculus in quasi-Banach space setting. More about this later. First we answer the well-posedness question that was solved using this stochastic calculus.

Theorem (Cioica, C., Veraar (2018))

Let

- $\alpha \in \mathbb{R}$, $p, q, r, T \in (0, \infty)$,
- $(K_t)_{t\in[0,\infty)}$ *the heat kernel,*
- $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ *a filtered probability space,*
- $W_n \colon [0, \infty) \times \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, *independent standard* $\mathbb{F}$*-Brownian motions, and*
- $g_n \colon [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ $\mathbb{F}$*-progressively measurable and such that* $\sup_{t\in[0,T]} \|(g_n)_{n\in\mathbb{N}}\|_{L^r(\Omega;B^{\alpha}_{p,q}(\mathbb{R}^d;L^2_{1/2-}(0,t;\ell^2)))} < \infty$.

Theorem (cont'd from previous slide)

For all $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $t \in [0,\infty)$ *define*

$$U(t)(\varphi) = (K_t * u_0)(\varphi) + \sum_{n=1}^{\infty} \int_0^t (K_{t-s} * g_n(s))(\varphi) \mathrm{d}W_n(s) \quad \mathbb{P}\text{-a.s.}$$

Then for all $\sigma \in [\alpha, \alpha+1)$, $u_0 \in L^r(\mathcal{F}_0; B^{\sigma}_{p,q}(\mathbb{R}^d))$ *the stochastic process* U *is well-defined as an* $\mathbb{F}$*-adapted,* $B^{\sigma}_{p,q}(\mathbb{R}^d)$*-valued process.*

Moreover, for all $\sigma \in [\alpha, \alpha+1)$*, all* $\lambda \in (0, \frac{1}{2}(\alpha+1-\sigma)] \cap (0, \frac{1}{2})$*, and all* $u_0 \in L^r(\mathcal{F}_0; B^{\sigma+2\lambda}_{p,q}(\mathbb{R}^d))$ *it holds that*

$$\|U\|_{C^{\lambda}([0,T];L^r(\Omega;B^{\sigma}_{p,q}(\mathbb{R}^d)))} < \infty. \tag{1}$$

Ingredients of proof

- Develop a stochastic calculus in a quasi-Banach space E:
  - introduce γ-radonifying operators $\gamma(H, E)$ (H is a Hilbert space);
  - introduce abstract stochastic integral for $R \in \gamma(H, E)$ with respect to an H-isonormal process W_H;
  - use decoupling techniques to identify spaces E for which $R \in L^r_{\mathbb{F}}(\Omega; \gamma(L^2(0, T; H); E))$ guarantees existence of an integral of R with respect to $W_{L^2(0,T;H)}$ (in this case E is said to *satisfy the decoupling property*.)

Ingredients of proof

- Develop a stochastic calculus in a quasi-Banach space E:
  - introduce γ-radonifying operators $\gamma(H, E)$ (H is a Hilbert space);
  - introduce abstract stochastic integral for $R \in \gamma(H, E)$ with respect to an H-isonormal process W_H;
  - use decoupling techniques to identify spaces E for which $R \in L^r_{\mathbb{F}}(\Omega; \gamma(L^2(0, T; H); E))$ guarantees existence of an integral of R with respect to $W_{L^2(0,T;H)}$ (in this case E is said to *satisfy the decoupling property*.)
- use 'standard' Fubini and Kahane-Khintchine arguments to show that $B^{\sigma}_{p,q}$ satisfies the decoupling property and to show that $\gamma(H, B^{\sigma}_{p,q}(\mathbb{R}^d)) \sim B^{\sigma}_{p,q}(\mathbb{R}^d, H)$.

Ingredients of proof

- Develop a stochastic calculus in a quasi-Banach space E:
  - introduce γ-radonifying operators $\gamma(H, E)$ (H is a Hilbert space);
  - introduce abstract stochastic integral for $R \in \gamma(H, E)$ with respect to an H-isonormal process W_H;
  - use decoupling techniques to identify spaces E for which $R \in L^r_{\mathbb{F}}(\Omega; \gamma(L^2(0, T; H); E))$ guarantees existence of an integral of R with respect to $W_{L^2(0,T;H)}$ (in this case E is said to *satisfy the decoupling property*.)
- use 'standard' Fubini and Kahane-Khintchine arguments to show that $B^{\sigma}_{p,q}$ satisfies the decoupling property and to show that $\gamma(H, B^{\sigma}_{p,q}(\mathbb{R}^d)) \sim B^{\sigma}_{p,q}(\mathbb{R}^d, H)$.
- verify that the heat kernel $(K_t)_{t\in[0,\infty)}$ has appropriate smoothing properties (using Fourier-multiplier techniques, here it is useful to work in the Besov space setting).

Differences between quasi-Banach and Banach space setting

Let E be an r-Banach space, $r \in (0,1)$, and let H be a Hilbert space.

1) For all $p \in [1,\infty)$, $x_1,\ldots,x_n \in E$, $(h_k)_{k=1}^n$ an ONS in H, $(\gamma_k)_{k=1}^n$ i.i.d. standard Gaussians it holds that

$$\left\| \sum_{k=1}^n \gamma_k x_k \right\|_{L^p(\Omega;E)} \leq \left\| \sum_{k=1}^n h_k \otimes x_k \right\|_{\gamma(H,E)} \leq 2^{\frac{1-r}{r}} \left\| \sum_{k=1}^n \gamma_k x_k \right\|_{L^p(\Omega;E)}.$$

Differences between quasi-Banach and Banach space setting (cont'd)

Let E be an r-Banach space, $r \in (0, 1)$, and let H be a Hilbert space.

2) The dual of E may be trivial (e.g. the dual of $L^p(0, 1)$ is trivial for $p \in (0, 1)$). Consequently, the stochastic integral cannot in general be identified by testing against the dual. Moreover, in general it is not clear whether the Karhunen-Loeve expansion exists.

Differences between quasi-Banach and Banach space setting (cont'd)

Let E be an r-Banach space, $r \in (0,1)$, and let H be a Hilbert space.

2) The dual of E may be trivial (e.g. the dual of $L^p(0,1)$ is trivial for $p \in (0,1)$). Consequently, the stochastic integral cannot in general be identified by testing against the dual. Moreover, in general it is not clear whether the Karhunen-Loeve expansion exists.

3) E does not satisfy the UMD property. Instead, we consider one-sided decoupling as in C. & Veraar (2010), G. & Geiss (2018) (this goes back to ideas of Kwapień and Woyczynski, and Garling, respectively).

Thank you for your attention!

Preprint: Cioica, Cox, and Veraar: “Stochastic integration in quasi-Banach spaces” available at arXiv:1804.08947