Para-Real Monte Carlo for American/Bermudan options

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The Parareal Method for an ODE

[J.-L. Lions-Maday-Turinici, 2001]. Parareal = Para-Real = Parallel + Real Time.

- Consider an ODE on $[0, T]$,

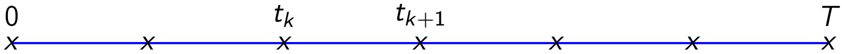
$$\dot{x} = b(t, x), \quad x(0) = x_0.$$

- To solve numerically this ODE, one introduces the Euler scheme with step $\Delta = \frac{T}{K}$, $K \in \mathbb{N}^*$, starting at x_0: let

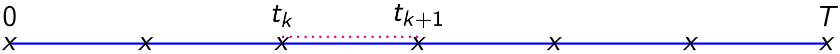
$$[0, T] = \cup_0^{K-1}[t_k, t_{k+1}], \quad t_k = \frac{kT}{K}.$$

and the standard Euler operator

$$x_{t_{k+1}} = \mathcal{E}_\Delta(x_{t_k}, t_k) := x_{t_k} + \Delta b(t_k, x_{t_k}), \quad k = 0 : K-1.$$



The Parareal Method for an ODE



- We divide each interval $[t_k, t_{k+1}]$ into smaller subintervals

$$[t_k, t_{k+1}] = \bigcup_{0}^{J-1} [t_{k,j}, t_{k,j+1}]$$

where we set $\delta = \frac{\Delta}{J} = \frac{T}{JK}$.

$$t_{k,j} = t_k + j\delta,\ j = 0 : J.$$

- Let $\mathcal{E}_\delta$ and $\mathcal{E}_\Delta$ be the Euler operators with time steps $\delta < \Delta$.
  Then, starting from $x_k \in \mathbb{R}^d$:
  - $\mathcal{E}_\delta^J(x_k, t_k) = \mathcal{E}_\delta(t_{k,J-1}, \cdot) \circ \cdots \circ \mathcal{E}(x_k, t_k)$ is the high precision solution at t_{k+1} starting from x_k at time t_k.
  - $\mathcal{E}_\Delta(x_k, t_k)$ is the low precision solution at t_{k+1} starting from x_k at t_k.

The Parareal Method for an ODE

- The Parareal scheme is an iteration loop over the forward loop in time:
  - Initialize: $x^0_{k+1} = x^0_k + \mathcal{E}_\Delta(x^0_k, t_k)$, $k = 0, \ldots, K-1$, $x^0_0 = x(0)$.
  - for $n = 0, \ldots, N-1$

    $x^{n+1}_0 = x(0)$.

    for $k = 0, \ldots, K-1$

    $$x^{n+1}_{k+1} = \underbrace{\mathcal{E}_\Delta(x^{n+1}_k, t_k)}_{low\ precision} + \underbrace{\mathcal{E}^J_\delta(x^n_k, t_k) - \mathcal{E}_\Delta(x^n_k, t_k)}_{high\ precision\ at\ n}.$$

    end (k)

    end (n).
- The coarse grid scheme is corrected by the error between the fine grid prediction and the the old coarse grid scheme computed at the former "old" value.
- For more see e.g. Martin Gander (SINUM, 2007).
- **Remark.** Note that x^n_k is the solution of the Euler scheme with fine step δ when as long as $k \le n$.

The Parareal Method for an ODE (parallel implementation)

- When the computation of the $(x_k^n)_{k=0:K}$ is completed (which is "fast" since Δ is "large").
- The K "refiners" $\big(\mathcal{E}_\delta^J(x_k^n, t_k) - \mathcal{E}_\Delta(x_k^n, t_k)\big)$, $k = 0 : K-1$ can be computed in parallel.
- For a fixed n, the global complexity is higher than a single fine Euler scheme (due to two coarse Euler schemes computations).
- but parallelization dramatically speeds up the execution (by almost a K factor).
- The parareal algorithm was devised for parallel architectures.
- A multilevel version can be derived by considering in cascade each fine level as a coarse level on which is defined a finer parareal scheme.

The Parareal Method for an SDE (I)

- Consider a d-dim Brownian diffusion process

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 \perp\!\!\!\perp W \; q\text{-S.B.M.}$$

with standard Lipschitz assumptions on $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and $\sigma[0, T] \times \mathbb{R}^d \to M(d, q, \mathbb{R})$;

- Replace mutatis mutandis the ODE Euler operator by the SDE Euler operator:

$$\mathcal{E}_\Delta(x, t, Z) = x + \Delta b(t, x) + \sqrt{\Delta}\,\sigma(t, x)Z, \quad Z \sim \mathcal{N}(0, I_q).$$

- As before, let $\delta = \frac{\Delta}{J}$, so that

$$[t_k, t_{k+1}] = \cup_{j=0}^{J-1}[t_{k,j}, t_{k,j+1}]$$

with

$$t_{k,j+1} = t_{k,j} + \delta, \; j = 0 : J-1 \;\text{ and }\; t_k = t_{k,0} = t_{k-1,J}.$$

Generate one/M path(s) of the Parareal scheme for an SDE

Initialization

- Generate i.i.d. $\mathcal{N}(0, I_q)$-distributed fine increments: $\big(Z_{k,j}\big)_{k=1:K, j=1:J}$.
- Coarse increments: Set $Z_k = \frac{Z_{k,1}+\cdots+Z_{k,J}}{\sqrt{J}}, \quad k = 1:K.$
- Initialize of the parareal scheme: $\widehat{X}^0_{t_{k+1}} = \mathcal{E}_\Delta(\widehat{X}^0_{t_k}, t_k, Z_k), \;\; k = 0 : K-1.$

for $n = 0 \; : \; N-1$ (parareal iterations)

for $k = 0 \; : \; K-1$ (forward time loop):

1. Fine grid solution $\{\widetilde{X}^{\delta,n}_{t_{k,j}}\}_{j=0}^J$ on $[t_k, t_{k+1}]$ with step δ, started at $t_{k,0} = t_k$ from $\widehat{X}^n_{t_k}$:

$$\widetilde{X}^{\delta,n}_{t_{k,0}} = \widehat{X}^n_{t_k}, \qquad \widetilde{X}^{\delta,n}_{t_{k,j+1}} = \mathcal{E}_\delta\big(\widetilde{X}^{\delta,n}_{t_{k,j}}, t_{k,j}, Z_{k,j+1}\big), \quad j = 0 : J-1.$$

2. Coarse grid solution at t_{k+1}: $\bar{X}^\Delta_{t_{k+1}} = \mathcal{E}_\Delta\big(\widehat{X}^{n+1}_{t_k}, t_k, Z_k\big)$.
3. Parareal updating: $\widehat{X}^{n+1}_{t_{k+1}} = \bar{X}^\Delta_{t_{k+1}} + \widetilde{X}^{\delta,n}_{t_{k,J}} - \widehat{X}^n_{t_{k+1}}$.

end k-loop.

end n-loop. [[For M paths repeat M times inside each loop (or //!)]].

The Parareal Method for an SDE: convergence rate I

Theorem 1 [P., Pironneau, Sall, '17, *SIFIN* '18] Let $n \leq K \in \mathbb{N}$.
Assume $(b, \sigma) \in C^0([0,T] \times \mathbb{R}^{d+d\otimes q})$, C^2 in x with Lipschitz continuous spatial derivatives, uniformly in $t \in [0,T]$. Let $(\bar{X}^{\delta}_{t_{k,j}})_{j,k}$ be the fine Euler scheme with step δ starting from $X_0 \in L^2(\mathbb{P})$.
There exists a real constant C only depending on T and the Lipschitz constants and norms of $b, b', b'', \sigma, \sigma', \sigma''$, such that:

- for $k \in \{n, \ldots, K\}$,

$$\|\widehat{X}^n_{t_k} - \bar{X}^{\delta}_{t_k}\|_{L^2(\mathbb{P})} \leq (C\Delta)^n \sqrt{\binom{k}{n}} \|\bar{X}^{\Delta}_{t_k} - \bar{X}^{\delta}_{t_k}\|_{L^2(\mathbb{P})}$$

$$\leq (C\Delta)^{n+\frac{1}{2}} \sqrt{\binom{k}{n}}.$$

- for all $0 \leq k \leq n$: $\widehat{X}^n_{t_k} = \bar{X}^{\delta}_{t_k}$ (coincide with the fine Euler scheme).

▷ Extends & improves [Bal-Maday, '04] (diffusion supposed to be simulable).

▷ Contains *ODE* error bounds when $\sigma \equiv 0$.

The Parareal Method for an SDE: convergence rate II

Theorem 2 [P., Pironneau, Sall, '17]

For fixed Δ, δ and n parareal iterations, the final and uniform errors satisfy

$$\max_{k=0:K} \|\widehat{X}^n_{t_k} - \bar{X}^\delta_{t_k}\|_{L^2(\mathbb{P})} \le (C\Delta)^{\frac{n+1}{2}} \sqrt{\binom{K}{n}}$$

$$\le \frac{(C\Delta)^{\frac{n+1}{2}}}{\sqrt{n!}} e^{-\frac{n(n-1)}{2}\frac{\Delta}{T}}$$

and

$$\Big\|\max_{k=0:K} |\widehat{X}^n_{t_k} - \bar{X}^\delta_{t_k}|\Big\|_{L^2(\mathbb{P})} \le \frac{(C'\Delta)^{\frac{n}{2}}}{\sqrt{(n+1)!}}\Big(1+\frac{\Delta}{T}\Big)^{\frac{1}{2}} e^{-\frac{n(n-1)}{2}\frac{\Delta}{T}}.$$

- Interchanging max and L^2-norm is costly: $(\widehat{X}^n_{t_k})_{k=0:K}$ is not a Markov chain.
- Unfortunately (?) for linear problems like computation of expectations, a naive parallelization ("path by path") is more efficient...
- Fortunately, not all problems in numerical probability are linear:

American/Bermudan Options & Tsitsiklis-Van Roy (& Longstaff-Schwartz)

- Dynamics: Brownian diffusion process or its Euler scheme.
- Bermudan/American payoff: $\varphi(x) = (K - x)^+$ the Put payoff, fair price := Snell envelope:

$$\mathcal{V}_t = \mathbb{P}\text{-esssup}\Big\{\mathbb{E}\big(\varphi(X_\tau)\,|\,\mathcal{F}_t^W\big),\ \tau : \Omega \to [t, T] \text{ stopping time }\Big\}$$

- Markov property: $\mathcal{V}_t = \nu(t, X_t)$, with v solution to the parabolic Variational Inequality:

$$\max(\partial_t \nu + L\nu, \varphi - \nu) = 0,\ t \in [0, T), \quad v(T, .) = \varphi$$

but if $d \geq 3$ or $4\ldots$

Time discretization

- Switch from $(X_t)_{t\in[0,T]}$ to the Euler schemes like $(\bar{X}_{t_k})_{k=0:K}$.
- Discrete time Snell envelope:

$$\mathcal{V}_{t_k} = \mathbb{P}\text{-esssup}\Big\{\mathbb{E}\big(\varphi(\bar{X}_\tau)\,|\,\mathcal{F}^W_{t_k}\big),\ \tau:\Omega\to [\![t_k,\ldots,t_K = T]\!] \text{ stopping time }\Big)\Big\}$$

- Markov property: $V_{t_k} = v(t_k, \bar{X}_{t_k}), \quad k = 0 : K.$
- Backward Dynamic Programming Principle: The $(\mathcal{F}_{t_k})_k$-Snell envelope satisfies

$$V_{t_n:=T} = \varphi(\bar{X}_T), \quad V_{t_k} = \max\big(\varphi(\bar{X}_{t_k}), C_{t_k}\big), \quad k = 0 : K-1,$$
$$\text{with} \quad C_{t_k} = \mathbb{E}[V_{t_{k+1}} \mid \mathcal{F}_{t_k}] = \mathbb{E}[V_{t_{k+1}} \mid X_{t_k}] = c_{t_k}(\bar{X}_{t_k})$$

Regression à la Titsiklis-Van Roy

- Projection on $\mathrm{span}\{\psi_\ell(\bar{X}_{t_k}),\ \ell = 1 : L\} \simeq L^2(\sigma(\bar{X}_{t_k}))$ (T.& van R.)

$$C_{t_k} \simeq \bar{c}(t_k, \bar{X}_{t_k}) := \sum_{j=1}^{J} \alpha^*_{k,j}\psi_j(\bar{X}_{t_k})$$

with $$\alpha^*_{k,\cdot} = \mathrm{argmin}_{\alpha\in\mathbb{R}^L}\mathbb{E}\left[\left(\bar{v}(t_{k+1}, \bar{X}_{t_{k+1}}) - \sum_{\ell=1}^{L}\alpha_{k,j}\psi_\ell(X_{t_k})\right)^2\right].$$

$$\bar{v}(t_k, \bar{X}_{t_k}) = \max\left(\varphi(\bar{X}_{t_k}), \bar{c}(t_k, \bar{X}_{t_k})\right).$$

- Explicit solution involving a Gram matrix, etc.
- $\bar{v}(0, x0) \simeq V_0$.

- Monte Carlo version: with form $\mathbb{P}_{X_{t_k}}$ to $\frac{1}{M}\sum_{m=1}^{M}\delta_{X_{t_k}^{(m)}}$ (empirical measure of i.i.d. copies).
- Implementation by Monte-Carlo means that the M paths speak to each other to perform the regression: It is a non linear problem.
- Hence naive parallelization is a problem!

Parareal Decomposition for American Options

initialization : Simulate $\widehat{V}_K^0(\omega) = \varphi(\widehat{X}_T^0(\omega))$, Compute

$$\widehat{V}_k^0 = \max\{\phi(\widehat{X}_{t_k}^0), \mathbb{E}[\widehat{V}_{k+1}^0 | \widehat{X}_{t_k}^0]\}, \; k = K-1 : 0.$$

for $n = 0 : N-1$ (parareal iterations) Compute $\bar{V}_K^{n+1} = \widehat{V}_K^{n+1} = \phi(\widehat{X}_T^{n+1})$

for $k = K-1, \ldots, 0$:

1. On each (t_k, t_{k+1}), from $\widetilde{V}_{k,J}^{\delta,n} = \mathbb{E}(\widehat{V}_{k+1}^n \,|\, \widetilde{X}_{k,J}^{\delta,n})$, compute

$$\widetilde{V}_{k,j}^{\delta,n} = \max\left\{\varphi(\widetilde{X}_{t_{k,j}}^{\delta,n}), \mathbb{E}[\widetilde{V}_{k,j+1}^{\delta,n} | \widetilde{X}_{t_{k,j}}^{\delta,n}]\right\}, \; j = J-1 \ldots 0.$$

2. Compute $\bar{V}_k^{n+1} = \max\left\{\varphi(\widehat{X}_{t_k}^{n+1}), \mathbb{E}[\bar{V}_{k+1}^{n+1} | \widehat{X}_{t_k}^{n+1}]\right\}$.
3. Set $\widehat{V}_k^{n+1} = \bar{V}_k^{n+1} + \widetilde{V}_{k,0}^{\delta,n} - \widehat{V}_k^n$.

end (backward) k-loop

end n-loop

With projections ($d = 1$ for simplicity)

Denote by $\mathfrak{P}f$ the projection of f on the monomials $1, x, \ldots, x^P$.

initialization : From $\widehat{V}^0_K(\omega^m) = \varphi(\widehat{X}^0_T(\omega^m))$, $m = 1 : M$ Compute

$$\widehat{V}^0_k = \max\{\phi(\widehat{X}^0_{t_k}), \mathfrak{P}\,\mathbb{E}[\widehat{V}^0_{k+1}|\widehat{X}^0_{t_k}]\},\ k = K-1 : 0.$$

for $n = 0 : N-1$ (parareal iterations) Compute $\bar{V}^{n+1}_K = \widehat{V}^{n+1}_K = \phi(\widehat{X}^{n+1}_T)$

for $k = K-1, \ldots, 0$:

1. On each (t_k, t_{k+1}), from $\widetilde{V}^{\delta,n}_{k,J} = \mathfrak{P}\,\mathbb{E}(\widehat{V}^n_{k+1}\,|\,\widetilde{X}^{\delta,n}_{k,J})$, compute

$$\widetilde{V}^{\delta,n}_{k,j} = \max\left\{\varphi(\widetilde{X}^{\delta,n}_{t_{k,j}}), \mathfrak{P}\,\mathbb{E}[\widetilde{V}^{\delta,n}_{k,j+1}|\widetilde{X}^{\delta,n}_{t_{k,j}}]\right\}, j = J-1 \ldots 0.$$

2. Compute $\bar{V}^{n+1}_k = \max\left\{\varphi(\widehat{X}^{n+1}_{t_k}), \mathfrak{P}\,\mathbb{E}[\bar{V}^{n+1}_{k+1}|\widehat{X}^{n+1}_{t_k}]\right\}$.
3. Set $\widehat{V}^{n+1}_k = \bar{V}^{n+1}_k + \widetilde{V}^{\delta,n}_{k,0} - \widehat{V}^n_k$.

end (backward) k-loop

end n-loop

Pros and cons

▷ Pros: All fine grid computations are local and can be assigned to separate processors: phase 1 in k-loop.

▷ Cons: $(\widehat{X}_k^n)_{k=0:K}$ is not a Markov chain: it keeps memory of the former iterations (in n).

- Hence

$$\bar{V}_K^{n+1} = \widehat{V}_K^{n+1} = \phi(\widehat{X}_T^{n+1}),\ \bar{V}_k^{n+1} = \max\left\{\varphi(\widehat{X}_{t_k}^{n+1}), \mathfrak{P}\,\mathbb{E}[\bar{V}_{k+1}^{n+1}|\widehat{X}_{t_k}^{n+1}]\right\}$$
$$k = K-1:0$$

  is not an $(\mathcal{F}_{t_k})_k$ Snell envelope... So we will be in theoretical trouble in the error analysis because of phase 2 in k-loop.

- This led us to introduce a variant

$$\widehat{V}_k^{n+1} = \bar{V}_k^{n+1} + \widetilde{V}_{k,0}^{\delta,n} - \bar{V}_k^n$$

  for which we could obtain theoretical results.

Parareal for American Option (II)

Proposition (True Snell envelopes) Let (for $k = 0 : K$)

$$\widehat{V}^{\Delta,n}_{t_k} = \mathbb{P}\text{-esssup}_{\tau\in\mathcal{T}^{\mathcal{F}}_{t_k}}\mathbb{E}[\varphi(\widehat{X}^n_\tau)|\mathcal{F}_{t_k}],\ \bar{V}^{\Delta,\delta}_{t_k} = \mathbb{P}\text{-esssup}_{\tau\in\mathcal{T}^{\mathcal{F}}_{t_k}}\mathbb{E}[\varphi(\widetilde{X}^\delta_\tau)|\mathcal{F}_{t_k}]$$

be the Snell envelopes of the parareal scheme $(\varphi(\widehat{X}^n_{t_k}))_{k=0:K}$ and of the fine Euler scheme $(\bar{X}^\delta_{t_k})_{k=0:K}$ observed at instants t_k.

$$\Big\| \max_{k=0,\dots,K} \big|\widehat{V}^{\Delta,n}_{t_k} - \bar{V}^{\Delta,\delta}_{t_k}\big|\Big\|_{L^2(\mathbb{P})} \le [\varphi]_{\rm Lip}\frac{(C\Delta)^{\frac{n}{2}}}{\sqrt{(n+1)!}}\left(1+\frac{\Delta}{T}\right)^{\frac12} e^{-\frac{n(n-1)}{2}\frac{\Delta}{T}}.$$

(Note that $(\bar{V}^{\Delta,\delta}_{t_k})_{k=0,\dots,K}$ is the coarse Snell envelope of the fine Euler scheme). At a fixed time t_k we have the better estimate

$$\big\|\widehat{V}^{\Delta,n}_{t_k} - \bar{V}^{\Delta,\delta}_{t_k}\big\|_2 \le [\varphi]_{Lip}(C\Delta)^n\sqrt{\binom{K+1}{n+1}-\binom{k}{n+1}}.$$

Forcing Markov property: pseudo-Snell envelopes

Proposition (Pseudo-Snell envelopes) Let (for $k = 0 : K$)

$$\widehat{V}^{\Delta,n}_{t_k} = \mathbb{P}\text{-}\mathrm{esssup}_{\tau\in\mathcal{T}^{\mathcal{F}}_{t_k}} \mathbb{E}[\varphi(\widehat{X}^n_\tau)|\widehat{X}_{t_k}], \quad \bar{V}^{\Delta,\delta}_{t_k} = \mathbb{P}\text{-}\mathrm{esssup}_{\tau\in\mathcal{T}^{\mathcal{F}}_{t_k}} \mathbb{E}[\varphi(\widetilde{X}^\delta_\tau)|\widehat{X}_{t_k}]$$

be the Pseudo-Snell envelopes of the parareal scheme $(\varphi(\widehat{X}^n_{t_k}))_{k=0:K}$ and of the fine Euler scheme $(\bar{X}^\delta_{t_k})_{k=0:K}$ observed at instants t_k.

$$\Big\| \max_{k=0,\ldots,K} \big| \widehat{V}^{\Delta,n}_{t_k} - \bar{V}^{\Delta,\delta}_{t_k} \big| \Big\|_{L^2(\mathbb{P})} \le [\varphi]_{\mathrm{Lip}} \frac{(C\Delta)^{\frac{n-1}{2}}}{\sqrt{(n+1)!}} \left(1 + \frac{\Delta}{T}\right)^{\frac{1}{2}} e^{-\frac{n(n-1)}{2}\frac{\Delta}{T}}.$$

Remark. The price to pay for non-Markovian feature of $(\widehat{X}^n_k)$ is higher.

Final result for the modified algorithm

We consider the modified parallel algorithm

$$\hat{V}_k^{n+1} = \bar{V}_{t_k}^{n+1} + \widetilde{V}_{t_{k,0}}^{\delta,n} - \bar{V}_{t_k}^{n},\ k =: K-1$$

where $\widehat{V}^0$ is the Snell envelope of the Euler scheme with step Δ.

Theorem

There exists a real constant $C = C_{b,\sigma,T}$ such that, for every $k = 0 : K-1$,

$$\max_{k=0:K} \left\| \hat{V}_{t_k}^{n+1} - \bar{V}_{t_k}^{\delta} \right\|_2 \leq [\varphi]_{\mathrm{Lip}} C\sqrt{\Delta}$$

Proof. The triangle inequality implies

$$\begin{aligned}\left\| \hat{V}_{t_k}^{n+1} - \bar{V}_{t_k}^{\delta} \right\|_2 &\leq \left\| \bar{V}_{t_k}^{n+1} - \widetilde{V}_{t_k}^{\delta,\Delta} \right\|_2 + \left\| \bar{V}_{t_{k,0}}^{\delta,n} - \bar{V}_{t_k}^{\delta} \right\|_2 + \left\| \bar{V}_{t_k}^{n} - \widetilde{V}_{t_k}^{\delta,\Delta} \right\|_2 \\ &\leq \frac{(C''\Delta)^{\frac{n}{2}}}{\sqrt{(n+2)!}}\left(1+\frac{\Delta}{T}\right)e^{-\frac{n(n+1)}{4}\frac{\Delta}{T}} + [f]_{\mathrm{Lip}} C\sqrt{\Delta} \\ &\quad + \frac{(C''\Delta)^{\frac{n-1}{2}}}{\sqrt{(n+1)!}}\left(1+\frac{\Delta}{T}\right)e^{-\frac{n(n-1)}{4}\frac{\Delta}{T}} \leq C_{b,\sigma,T}[f]_{\mathrm{Lip}}\Delta\end{aligned}$$

Results (I): The Black-Scholes Case

▷ Underlying asset is given by the Black-Scholes *SDE*,

$$b(x,t) = rx \quad \sigma(x,t) = \sigma_0 x \quad \text{with} \quad r = 0.05,\ \sigma_0 = 0.2$$

and $\varphi(x) = (x_\kappa)_+$

$$x_0 = 36,\ \kappa = 40,\ T = 2.$$

▷ True price $= 4.478$ (by VI-PDE with finite difference).

▷ Projection is performed on $\{1, x, x^2\}$ and the Monte Carlo froward simulation with $M = 100\,000$ paths.

▷ We implemented the "natural" parareal algorithm with a Tsitsiklis-Van Roy algorithm (BDPP on the continuation function)

$$\hat{V}_k^{n+1} = \bar{V}_{t_k}^{n+1} + \widetilde{V}_{t_{k,0}}^{\delta,n} - \widehat{V}_{t_k}^{n},\ k =: K-1$$

▷ We chose a constant fine grid with $\delta = T/32$. Free parameters are Δ (i.e. the number of points on the coarse grid) and n the number of parareal iterations.

Results (Ia): The Black-Scholes Case

K	J	Δ	$n=1$	$n=2$	$n=3$	$n=4$
2	16	0.666667	0.60338	0.152339	0.0171122	0.000833293
4	8	0.4	0.237451	0.0437726	0.00217885	0.000725382
8	4	0.222222	0.0854814	0.0156243	0.000735309	0.000515332
16	2	0.117647	0.0257407	0.00120513	0.000439038	0.000262921
2	16	0.666667	0.5912463	0.1434691	0.0418341	0.0414722
4	8	0.4	0.2245711	0.0743709	0.0225051	0.0224303
8	4	0.222222	0.0740923	0.0205441	0.0072178	0.0072066
16	2	0.117647	0.0194701	0.0021758	0.0021592	0.0021509

Table: *Absolute error from the American payoff computed on the fine grid by a sequential LSMC Tsitsikli-Van Roy algorithm and the same computed using the parareal iterative algorithms (Top: TLPRAO vs Bottom:TLPRAOA). The coarse grid has K intervals; the coarse time step is Δ/K; the fine grid has a fixed number of points hence each interval $(t_k, t_{t_{k+1}})$ it has J sub-intervals.*

Remark. True Longstaff-Schwartz *LSMC* algorithm based on running optimal stopping times yields similar results.

Results (Ib): The Black-Scholes Case

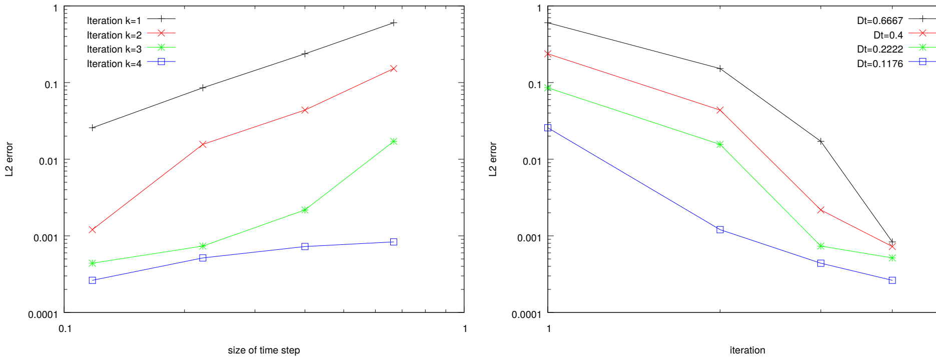
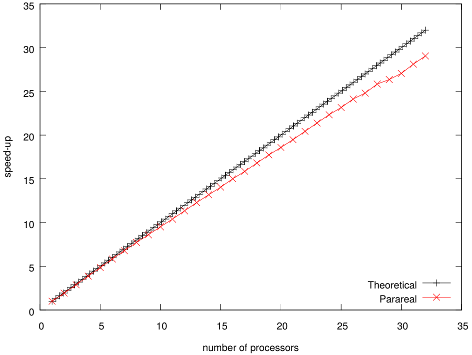


Figure: *Black-Scholes case: Errors on the payoff versus Δ on the left for several values of n and versus n on the right for several values of Δ. Both graphs are for Algorithm* **??** *in log-log scales and indicate a general behavior of the error ϵ not incompatible with (1).*

Results(II): speed-up induced by parallel implementation of the parallel method



Speed-up versus the number of processors, i.e. the parareal CPU time on a parallel machine divided by the parareal CPU time on the same machine but running on one processor. There are two levels only; the parameters are $N_{proc} = 1, 2, \ldots, 32$, $n = 2$ and $J = 100$ so as to keep each processor fully busy.

Results (III): CEV model

The diffusion coefficient now depends on the price of the risky asset: $\sigma(x,t) = \sigma_0 x^{0.7}$ (i.e. the volatility itself is given by $\sigma_0 x^{-0.3}$). All parameters have the same values as above.

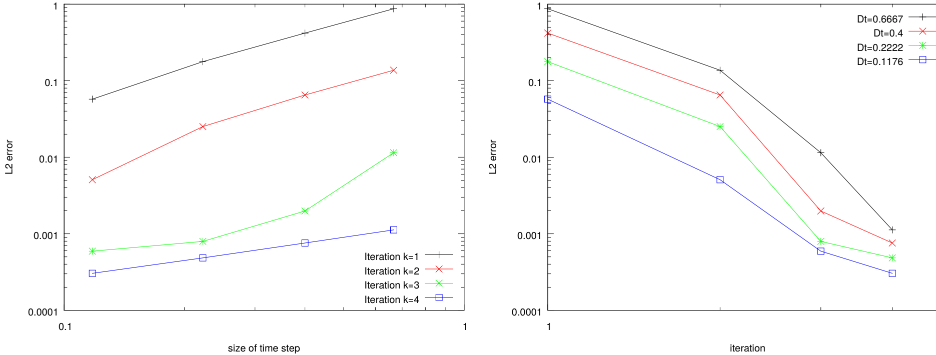


Figure: *Constant Elasticity case.* Left: *Errors on the price vs* Δ *on the left for several values of* n. Right: *versus* n *on the right for several values of* Δ. *Both graphs are for Algo. TLPRAO in log-log scales and indicate a general behavior of the error* ϵ *not incompatible with (1).*