# Quasi-Monte Carlo integration over a triangle

Takashi Goda

School of Engineering, University of Tokyo

Joint work with Kosuke Suzuki and Takehito Yoshiki

MCQMC conference, July 2018

# Integration over $\triangle$

- Let $\triangle \subset \mathbb{R}^2$ be a triangle with Lebesgue measure denoted by $|\triangle|$.
- For an integrable $f\colon \triangle \to \mathbb{R}$, define

$$I(f) := \frac{1}{|\triangle|} \int_{\triangle} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

- We want to approximate $I(f)$ using function evaluations.
- Such problem arises in graphics rendering and finite element methods.

# Quadrature over △

- Consider a linear algorithm

$$A_N(f) = \sum_{n=0}^{N-1} w_n f(\mathbf{x}_n).$$

The Gauss-Legendre quadrature is popular for smooth $f$.

In this work we consider Quasi-Monte Carlo (QMC) integration

$$I_P(f) := \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n),$$

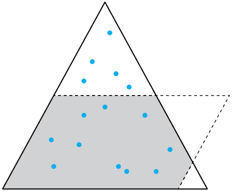
and give an explicit construction of $P$ for twice differentiable $f$.

# Koksma-Hlawka

- Brandolini, Colzani, Gigante & Travaglini (2013) prove a Koksma-Hlawka type inequality for $\triangle$ (more generally for simplices):

$$|I(f) - I_P(f)| \leq D^*(P)V(f).$$

- The discrepancy $D^*(P)$ is the supremum of



among three vertices (set as origin).

# Review of Basu & Owen (2015)

- Our work is inspired by their one.
- They come up with two constructions of low-$D^*(P)$ point sets in $\triangle$:

1. Triangular van der Corput sequence:

$$D^*(P) = \frac{2}{3\sqrt{N}} - \frac{1}{9N}$$

for $N = 4^\ell$, extensible, and can be randomized.

2. Triangular Kronecker lattices (rotation of an integer lattice through an angle whose tangent is badly approximable):

$$D^*(P) \leq C\frac{\log N}{N}.$$

# van der Corput (in base 4)

- Denote the 4-adic expansion of $k \in \mathbb{N}_0$ by

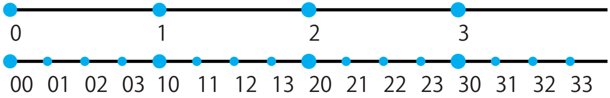
$$k = (\ldots \kappa_2 \kappa_1 \kappa_0)_4$$

where all but a finite number of $\kappa_i$'s are 0.

- The original van der Corput sequence is given by $\{\phi(k) : k \in \mathbb{N}_0\}$

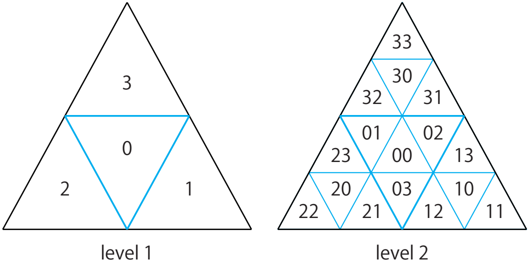
$$\phi(k) = (0.\kappa_0 \kappa_1 \kappa_2 \ldots)_4$$

- This is based on a recursive partitioning of the unit interval:



# Triangular van der Corput

- Replace the recursive partitioning with the following one:




- $\kappa_0$ determines which subtriangle $k$ is mapped to,
  $\kappa_1$ does which sub-subtriangle $k$ is mapped to, and so on.
- "Whether the subregion of level $\ell$ is $\triangle$ or $\nabla$" is same as whether

$$\sigma_\ell(k) := \#\{0 \leq i \leq \ell - 1 : \kappa_i = 0\}$$

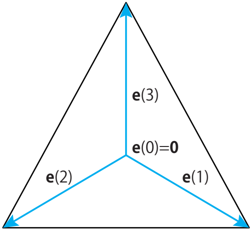
is even ($\triangle$) or odd ($\nabla$).

# Triangular van der Corput (cont'd)

- This idea gives the mapping

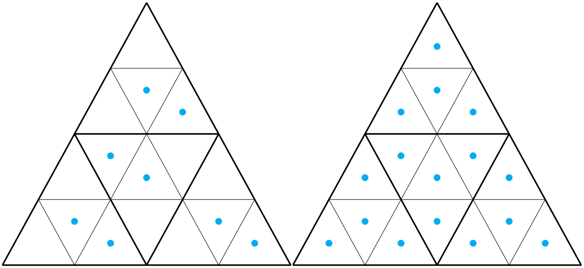
$$\tau(k) = \frac{\mathbf{e}(\kappa_0)}{2} + (-1)^{\sigma_1(k)} \frac{\mathbf{e}(\kappa_1)}{2^2} + (-1)^{\sigma_2(k)} \frac{\mathbf{e}(\kappa_2)}{2^3} + \cdots$$

  where




- The *triangular* van der Corput sequence is given by $\{\tau(k) : k \in \mathbb{N}_0\}$.

# Triangular van der Corput (cont'd)



$N = 8$ $\qquad$ $N = 16$

- For $N = 4^\ell$, every subregion of level $\ell$ has exactly one point in its center, giving

$$D^*(P) = \frac{2}{3\sqrt{N}} - \frac{1}{9N}$$

# Our idea

1. Since we are dealing with 2D domains, it is natural to consider 2D QMC points or sequences instead of 1D van der Corput sequence.

2. Focusing on smooth functions, there is a chance to conduct Walsh analysis as done in the unit cube setting (Dick, 2009; Yoshiki, 2017).

3. Given a decay of Walsh coefficients, it might be possible to prove the worst-case error bound.

# Class of functions and worst-case error

- In this work, we consider twice differentiable $f$ with the norm

$$\|f\|_{C^2(\triangle)} := \max_{0\le \delta_1+\delta_2\le 2} \left\| \frac{\partial^{\delta_1+\delta_2} f}{\partial^{\delta_1} x_1 \partial^{\delta_2} x_2} \right\|_{L^\infty(\triangle)} .$$

- The worst-case error is defined by

$$e^{\mathrm{wor}}(C^2(\triangle); A_N) = \sup_{\|f\|_{C^2(\triangle)}\le 1} |I(f) - A_N(f)| .$$

For QMC integration using a point set $P$, $A_N$ is replaced by $I_P$.

# Digital sequences (in base 2)

- Let $C_1, C_2 \in \mathbb{F}_2^{\mathbb{N}\times\mathbb{N}}$, where $\mathbb{F}_2$ denotes the two-element field.
- Denote the dyadic expansion of $k \in \mathbb{N}_0$ by

$$k = (\ldots \kappa_2 \kappa_1 \kappa_0)_2$$

  where all but a finite number of $\kappa_i$'s are 0.
- Let

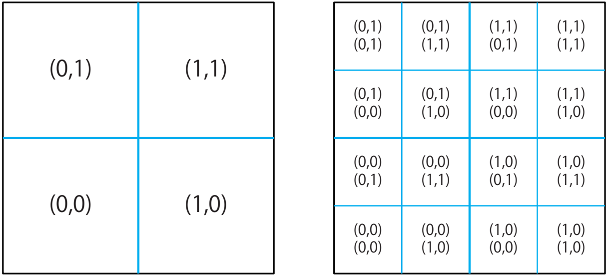
$$\begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \\ \vdots \end{pmatrix} = C_1 \begin{pmatrix} \kappa_0 \\ \kappa_1 \\ \vdots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \\ \vdots \end{pmatrix} = C_2 \begin{pmatrix} \kappa_0 \\ \kappa_1 \\ \vdots \end{pmatrix}.$$

- The digital sequence with $C_1, C_2$ is given by

$$\left\{ ((0.\xi_1^{(1)}\xi_2^{(1)}\ldots)_2, (0.\xi_1^{(2)}\xi_2^{(2)}\ldots)_2) : k \in \mathbb{N}_0 \right\}.$$

# Recursive partitioning

- Digital sequences are based on a recursive partitioning of $\square$:




level 1 level 2

- $(\xi_1^{(1)}, \xi_1^{(2)})$ determines which subcube $k$ is mapped to,
  $(\xi_2^{(1)}, \xi_2^{(2)})$ does which sub-subcube $k$ is mapped to, and so on.

# Quality criterion of $C_1, C_2$ for $\square$

- For simplicity, consider $C_1, C_2 \in \mathbb{F}_2^{\ell\times\ell}$ for finite $\ell$. Define

$$P^\perp := \left\{ K = (\kappa_{ij}) \in \mathbb{F}_2^{\ell\times 2} : C_1^\top \begin{pmatrix} \kappa_{11} \\ \vdots \\ \kappa_{\ell 1} \end{pmatrix} \oplus C_2^\top \begin{pmatrix} \kappa_{12} \\ \vdots \\ \kappa_{\ell 2} \end{pmatrix} = \mathbf{0} \right\}.$$

- For $k = (\kappa_i) \in \mathbb{F}_2^\ell$, define

$$\mu_\square(k) = \begin{cases} 0 & \text{if } k = \mathbf{0}, \\ \max\{i \le \ell : \kappa_i \neq 0\} & \text{otherwise.} \end{cases}$$

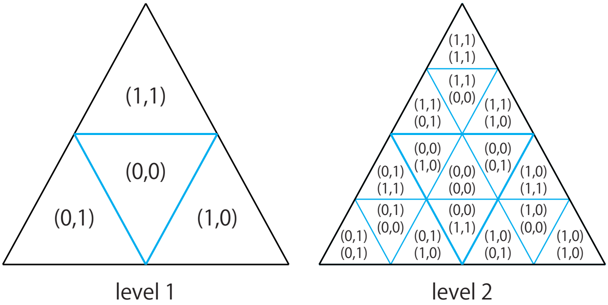
  For $K = (k_1, k_2) \in \mathbb{F}_2^{\ell\times 2}$, $\mu_\square(K) := \mu_\square(k_1) + \mu_\square(k_2)$.
- So-called $t$-value is defined as the minimum $t$ such that

$$\min_{K\in P^\perp\setminus\{\mathbf{0}\}} \mu_\square(K) \ge \ell - t + 1.$$

- Construction of $C_1, C_2$ with small $t$-value: Sobol' (1967), Faure (1982), Niederreiter (1988), etc.

# Triangular digital sequences

- Replace the recursive partitioning with the following one:




- "Whether the subregion of level $\ell$ is $\triangle$ or $\nabla$" is same as whether

$$\sigma_\ell(k) := \#\{1 \le i \le \ell : (\xi_i^{(1)}, \xi_i^{(2)}) = (0,0)\}$$

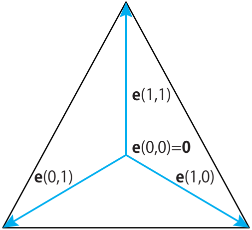
is even ($\triangle$) or odd ($\nabla$).

# Triangular digital sequences (cont'd)

- The *triangular* digital sequence with $C_1, C_2$ is $\{\tau(k) : k \in \mathbb{N}_0\}$ with

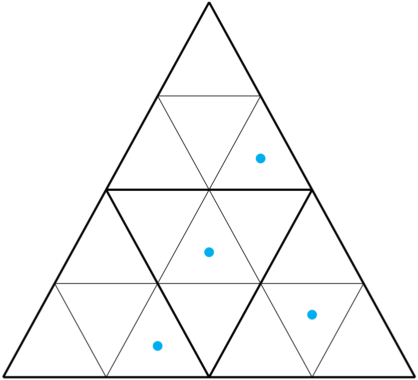
$$\tau(k) = \frac{\mathbf{e}(\xi_1^{(1)}, \xi_1^{(2)})}{2} + (-1)^{\sigma_1(k)} \frac{\mathbf{e}(\xi_2^{(1)}, \xi_2^{(2)})}{2^2} + (-1)^{\sigma_2(k)} \frac{\mathbf{e}(\xi_3^{(1)}, \xi_3^{(2)})}{2^3} + \cdots$$

where



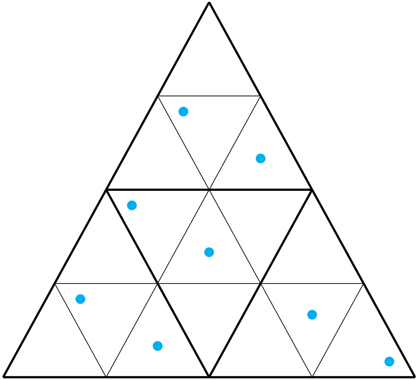

- The triangular regular grids is same as the triangular van der Corput.
  $\rightarrow$ Our construction includes Basu-Owen's one as a special case.

# Triangular Sobol' sequences



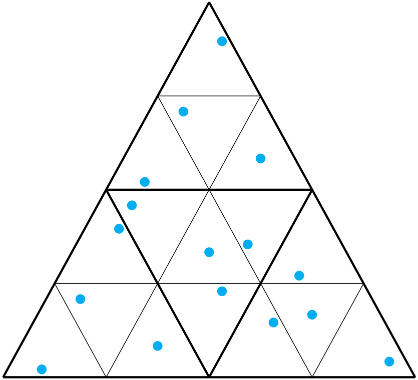
$N = 4$

# Triangular Sobol' sequences



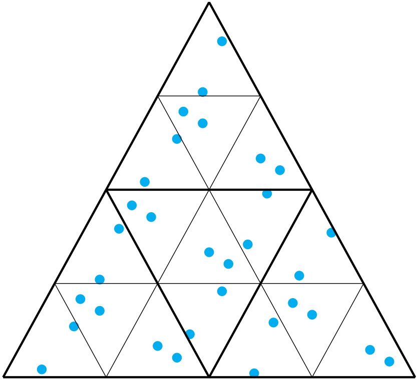
$N = 8$

# Triangular Sobol' sequences



$N = 16$

# Triangular Sobol' sequences



$N = 32$

# Discretization

- For a fixed level $\ell$, the set of $4^\ell$ subregions and $\mathbb{F}_2^{\ell\times 2}$ are one-to-one.
- Denote the subregion corresponding to $X \in \mathbb{F}_2^{\ell\times 2}$ by $\triangle_\ell(X)$. Define

$$f_\ell(X) = \frac{1}{4^\ell|\triangle|}\int_{\triangle_\ell(X)} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

**Lemma**

*For $X \in \mathbb{F}_2^{\ell\times 2}$ and $\mathbf{x} \in \triangle_\ell(X)$, we have*

$$|f(\mathbf{x}) - f_\ell(X)| \le \frac{\sqrt{2}d(\triangle)\|f\|_{C^2(\triangle)}}{2^\ell}$$

*where $d(\triangle)$ denotes the diameter of $\triangle$.*

# Walsh series

- For $X = (\xi_{ij}), K = (\kappa_{ij}) \in \mathbb{F}_2^{\ell \times 2}$, define

$$\mathrm{wal}_K(X) := (-1)^{\sum_{i,j} \kappa_{ij}\xi_{ij}} = \prod_{i,j} (-1)^{\kappa_{ij}\xi_{ij}},$$

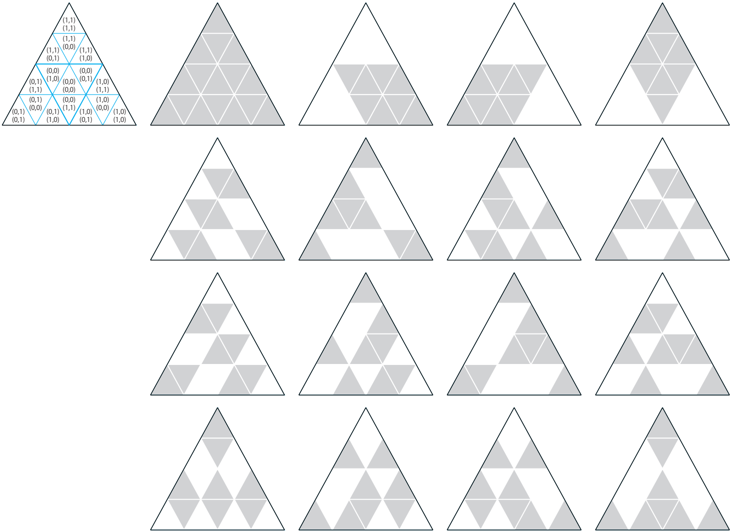
  called the $K$-th Walsh function.
- For $f_\ell \colon \mathbb{F}_2^{\ell \times 2} \to \mathbb{R}$, we have a Walsh series expansion:

$$f_\ell(X) = \sum_{K \in \mathbb{F}_2^{\ell \times 2}} \hat{f}_\ell(K) \mathrm{wal}_K(X)$$

  where

$$\hat{f}_\ell(K) = \frac{1}{4^\ell} \sum_{X \in \mathbb{F}_2^{\ell \times 2}} f_\ell(X) \mathrm{wal}_K(X).$$

# Triangular Walsh functions



# Decay of Walsh coefficients

- For $K = (k_1, k_2) \in \mathbb{F}_2^{\ell \times 2}$, define
$$\mu_\triangle(K) = \max\{\mu_\square(k_1), \mu_\square(k_2)\}.$$
(Recall: $\mu_\square(K) := \mu_\square(k_1) + \mu_\square(k_2)$)

**Lemma**

*For $\ell \in \mathbb{N}$ and $K \in \mathbb{F}_2^{\ell \times 2}$, we have*
$$|\hat{f}_\ell(K)| \leq \max\left(2\sqrt{2}d(\triangle), 4d^2(\triangle)\right) \|f\|_{C^2(\triangle)} \frac{\mu_\triangle(K)}{2^{\mu_\triangle(K)}}.$$

# Quality criterion of $C_1, C_2$ for $\triangle$

- $C_1, C_2$ with small $t$-value is already OK because

$$\min_{K \in P^{\perp} \setminus \{\mathbf{0}\}} \mu_{\triangle}(K) \geq \min_{K \in P^{\perp} \setminus \{\mathbf{0}\}} \frac{\mu_{\square}(K)}{2} \geq \frac{\ell - t + 1}{2}$$

- The triangular regular grids is also good in this regard. Thus the triangular van der Corput sequence due to Basu & Owen (2015) is not optimal in terms of $D^*(P)$, but almost optimal for numerical integration of $f \in C^2(\triangle)$ (as shown below).

# Upper bound

- The following is the main result of this work:

**Theorem (G., Suzuki and Yoshiki, 2017)**

*QMC integration using a triangular digital sequence with large minimum $\mu_\triangle$-value achieves*

$$e^{\mathrm{wor}}(C^2(\triangle); I_P) \leq C\frac{(\log N)^3}{N}.$$

*Morevoer, for $N = 2^\ell$ with $\ell \in \mathbb{N}$, we have*

$$e^{\mathrm{wor}}(C^2(\triangle); I_P) \leq C\frac{\ell^2}{2^\ell}.$$

# Lower bound

- The bump function technique from Bakhvalov (1959) gives:

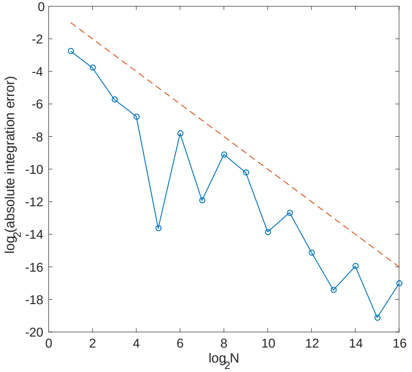
**Theorem**

*For any $A_N$, there exists a constant $c > 0$ s.t.*

$$e^{\mathrm{wor}}(C^2(\triangle); A_N) \geq \frac{c}{N}.$$

# Numerics

- Consider $\triangle = \{(x, y) : x, y \geq 0, x + y \leq 1\}$ and use triangular Sobol' sequences.
- The test function is $f(x, y) = e^x$ for which we have $I(f) = 2(e - 2)$.
- The red line shows $N^{-1}$ as a reference.



Thank you for your attention!