Approximation rate of BSDEs using random walk

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Introduction

Numerical scheme and convergence results

Numerical Examples

Generalization to a diffusion process
Random walk approximation of the Brownian motion

Let \( t_k := kh, k = 0, \cdots, n \) be a regular grid of \([0, T]\), where \( h = \frac{T}{n} \) and define

\[
B^n_t := \sqrt{h} \sum_{k=1}^{\lceil t/h \rceil} \varepsilon_k, \quad (\varepsilon_k)_{k=1, \cdots, n} \text{ i.i.d. Bernoulli r.v.}:
\]

\[
\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}
\]

- Donsker’s Theorem:

\[
(B^n_t)_{t \in [0, T]} \rightarrow (B_t)_{t \in [0, T]} \quad \text{in distribution.}
\]

- Wanted:

\[
B^n_t \rightarrow B_t \quad \text{in } L_2 \quad \text{for all } t \in [0, T] \text{ with a convergence rate.}
\]
Random walk constructed by Skorohod embedding

\[ \tau_0 = 0, \quad \tau_k := \inf\{ t > \tau_{k-1} : |B_t - B_{\tau_{k-1}}| = \sqrt{h} \} \]
Properties of \((B_{\tau_k})_k\) and \((\tau_k)_k\)

- \((B_{\tau_k} - B_{\tau_{k-1}})_{k \geq 1} \overset{d}{=} (\sqrt{h} \varepsilon_k)_{k \geq 1}\)
- \((\tau_k - \tau_{k-1})_k\) are i.i.d. \(\sim \tau := \inf\{ t > 0 : |B_t| = \sqrt{h} \}\)
- It holds \(\mathbb{E}(\tau_k) = t_k, \quad \mathbb{E}|\tau_k - t_k|^2 = \frac{2}{3} t_k h, \quad k = 1, \ldots, n.\)

\[
\mathbb{E}|B_{\tau_k} - B_{t_k}|^2 = \mathbb{E}|\tau_k - t_k| \leq \sqrt{\frac{2}{3} t_k h}
\]

- random walk constructed from B: \(B^n_t := \sqrt{h} \sum_{k=1}^{[t/h]} (B_{\tau_k} - B_{\tau_{k-1}})\)

Since \(B^n_{t_k} = B_{\tau_k}\) we have for \(t \in [t_k, t_{k+1})\)

\[
\mathbb{E}|B^n_t - B_t|^2 \leq 2\mathbb{E}|B_{\tau_k} - B_{t_k}|^2 + 2\mathbb{E}|B_{t_k} - B_t|^2 \\
\quad \leq 2\sqrt{\frac{2}{3} t_k \frac{T}{n}} + 2 \frac{T}{n} \leq 4 \frac{T}{\sqrt{n}}
\]
Forward Backward Stochastic Differential Equations

\[
X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dB_r, \quad 0 \leq t \leq T
\]

\[
Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_sdB_s
\]

replace \((B_t)_{t \in [0, T]}\) by a random walk \((B^n_t)_{t \in [0, T]}\) what kind of convergence

\[
((X^n_t)_{t \in [0, T]}, (Y^n_t)_{t \in [0, T]}, (Z^n_t)_{t \in [0, T]}) \rightarrow ((X_t)_{t \in [0, T]}, (Y_t)_{t \in [0, T]}, (Z_t)_{t \in [0, T]})
\]

Briand, Delyon and Mémin (2001)

If \(b, \sigma, f\) and \(g\) are Lipschitz and \((B^n_t)_{t \in [0, T]}\) such that

\[
\sup_{0 \leq t \leq T} |B^n_t - B_t| \rightarrow 0, \quad n \rightarrow \infty, \quad \text{in probability, then}
\]

\[
\sup_{0 \leq t \leq T} |Y^n_t - Y_t|^2 + \int_0^T |Z^n_s - Z_s|^2 ds \rightarrow 0 \quad \text{when } n \rightarrow \infty \quad \text{in probability.}
\]
Other results

- Toldo (2005) extends the previous result to BSDEs with random terminal time

$\Rightarrow$ convergence in probability or weak convergence: no rate
Introduction

Numerical scheme and convergence results

Numerical Examples

Generalization to a diffusion process
Discretization

We consider the BSDE

\[ Y_t = g(B_T) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T \]

and its approximation

\[ Y^n_{t_k} = g(B^n_T) + h \sum_{m=k}^{n-1} f(t_{m+1}, Y^n_m, Z^n_m) - \sum_{m=k}^{n-1} Z^n_m (B^n_{m+1} - B^n_m). \]

For \( n \) large enough: \( \exists! (\mathcal{F}_{\tau_k})_k \)-adapted solution \((Y^n_{t_k}, Z^n_{t_k})_{k=0}^{n-1}\).

The corresponding scheme:

\[ Y^n_{t_n} := g \left( \sum_{k=1}^{n} \sqrt{h\varepsilon_k} \right) \]

\[ Z^n_{t_{k-1}} := \frac{\mathbb{E}_{\tau_{k-1}}[Y^n_{t_k} | \varepsilon_k]}{\sqrt{h}}, \quad k = 1, \ldots, n \]

\[ Y^n_{t_{k-1}} := \mathbb{E}_{\tau_{k-1}}[Y^n_{t_k}] + f(t_k, Y^n_{t_{k-1}}, Z^n_{t_{k-1}})h \]
Discretization: approximation of $Z$

From Zhang’s paper (2005), we have

$$Z_t = \mathbb{E}_t \left( g(B_T) \frac{B_T - B_t}{T - t} \right) + \mathbb{E}_t \left( \int_t^T f(s, Y_s, Z_s) \frac{B_s - B_t}{s - t} \, ds \right),$$

**Lemma**

For all $k = 0, 1, \ldots, n - 1$, we have

$$Z^n_{t_k} = \mathbb{E}_{\tau_k} \left( g(B^n_T) \frac{B^n_{t_n} - B^n_{t_k}}{t_n - t_k} \right) + h\mathbb{E}_{\tau_k} \left( \sum_{m=k+1}^{n-1} f(t_{m+1}, Y^n_{t_m}, Z^n_{t_m}) \frac{B^n_{t_m} - B^n_{t_k}}{t_m - t_k} \right).$$
Convergence results

Assume: \( \forall (t, y, z), (t', y', z') \in [0, T] \times \mathbb{R}^2 \),

\[
|f(t, y, z) - f(t', y', z')| \leq L_f (\sqrt{t - t'} + |y - y'| + |z - z'|),
\]

\( \forall (x, x') \in \mathbb{R}^2, \quad |g(x) - g(x')| \leq C_g (1 + |x|^{p_0} + |x'|^{p_0}) |x - x'|^\alpha. \)

Theorem (C.G., Labart, Luoto)

\[
\sup_{0 \leq t < T} \mathbb{E} |Y_t - Y^n_t|^2 \leq C_1 h^\frac{\alpha}{2},
\]

\[
\mathbb{E} \int_0^T |Z_t - Z^n_t|^2 dt \leq C_2 h^\beta \quad \text{for } \beta \in (0, \frac{\alpha}{2})
\]

where \( C_1 \) depends on \( L_f, T, C_g, p_0 \) and \( \alpha \) and \( C_2 \) additionally on \( \beta \).

The reason for \( \beta \):

\[
\mathbb{E} |Z_t - Z^n_t|^2 \leq C_0 \frac{h^\frac{\alpha}{2}}{T-t_k} + C_1 \frac{h^\frac{\alpha}{2}}{(T-t)^{1-\frac{\alpha}{2}}} 1_{t \neq t_k} \quad \text{for } t \in [t_k, t_{k+1}), \; k = 0, \ldots, n - 1,
\]
Introduction

Numerical scheme and convergence results

Numerical Examples

Generalization to a diffusion process
Example 1

\[ dY_t = -(Y_t + Z_t) dt + Z_t dB_t, \]
\[ Y_T = B_T^2 \]

The explicit solution is given by

\[ Y_t = e^{T-t}((B_t + T-t)^2 + T-t), \quad Z_t = 2e^{T-t}(B_t + T-t) \]

We compute by Monte Carlo \( \mathbb{E}(|Y_t - Y_t^n|^2) \) and \( \mathbb{E}(|Z_t - Z_t^n|^2) \) for different values of \( n \).

\( g \) is a locally Lipschitz function : the convergence should go faster than \( \frac{1}{\sqrt{n}} \).
Figure: log(Error on $Y$) for different values of log($n$)

We get a slope of $-0.46$
Example 1: error on $Z$

Figure: $\log(\text{Error on } Z)$ for different values of $\log(n)$

We get a slope of $-0.48$
Example 2

\[ dY_t = -(Y_t + Z_t)dt + Z_t dB_t, \]
\[ Y_T = \sqrt{|B_T|} \]

The solution is given by

\[ Y_t = e^{\frac{T-t}{2}} \tilde{E}(\sqrt{|\tilde{B}_{T-t} + B_t|} e^{\tilde{B}_{T-t}}) \]

We compute by Monte Carlo \( E(|Y_t - Y_t^n|^2) \) and \( E(|Z_t - Z_t^n|^2) \) for different values of \( n \).

\( g \) is a \( \frac{1}{2} \)-Hölder function : the convergence should go faster \( \frac{1}{n^4} \).
Example 2: error on $Y$

Figure: $\log(\text{Error on } Y)$ for different values of $\log(n)$

We get a slope of $-0.56!!!!$
Introduction

Numerical scheme and convergence results

Numerical Examples

Generalization to a diffusion process
The general case

\[ X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dB_r, \quad 0 \leq t \leq T \]

\[ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \]

- \( b, \sigma \in C_b^{0,2} \) and \( \frac{1}{2} \)-Hölder in time, unif. in space
- the first and second derivatives w.r.t. the space variable are assumed to be \( \gamma \)-Hölder continuous (for some \( \gamma \in ]0, 1] \), w.r.t. the parabolic metric \( d((x, t), (x', t')) = (|x - x'|^2 + |t - t'|)^{\frac{1}{2}} \) on all compact subsets of \([0, T] \times \mathbb{R}\).
- \( \sigma(t, x) \geq \delta > 0 \)
Numerical Scheme for \( X \) and \( Y \)

\[
X_0^n = x,
\]

\[
X_{tk}^n = x + \sum_{j=1}^{k} b(t_j, X_{t_{j-1}}^n)h + \sqrt{h} \sum_{j=1}^{k} \sigma(t_j, X_{t_{j-1}}^n) \varepsilon_j
\]

\[
Y_{tk}^n = g(X_T^n) + h \sum_{m=k}^{n-1} f(t_{m+1}, X_{tm}^n, Y_{tm}^n, Z_{tm}^n) - \sqrt{h} \sum_{m=k}^{n-1} Z_{tm}^n \varepsilon_{m+1}.
\]

and

\[
Y_{tk}^n = \mathbb{E}_{\tau_k} \left( g(X_{tn}^n) + h \sum_{m=k}^{n-1} f(t_{m+1}, X_{tm}^n, Y_{tm}^n, Z_{tm}^n) \right)
\]

\[
Z_{tk}^n = \frac{\mathbb{E}_{\tau_k} g(X_{tn}^n) \varepsilon_{k+1}}{\sqrt{h}} + \mathbb{E}_{\tau_k} \left( \sqrt{h} \sum_{m=k+1}^{n-1} f(t_{m+1}, X_{tm}^n, Y_{tm}^n, Z_{tm}^n) \varepsilon_{k+1} \right)
\]
Approximation of $Z$

$$Z_t = \mathbb{E}_t \left( g(X_T) N_t^T + \int_t^T f(r, X_r, Y_r, Z_r) N_r^t dr \right) \sigma(t, X_t), \quad \text{Zhang (2005)}$$

where $N_v^t := \frac{1}{V - t} \int_t^V \frac{\nabla X_s}{\sigma(s, X_s) \nabla X_t} dB_s$ is the Malliavin weight of first order

$$\hat{Z}_{tk}^n := \mathbb{E}_{\tau_k} \left( g(X_T^n) N_{\tau_k}^{n, t_n} + h \sum_{m=k+1}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) N_{\tau_m}^{n, t_m} \right),$$

$$N_{\tau_m}^{n, t_m} = \frac{1}{t_m - t_k} \sum_{l=k+1}^{m} \frac{\nabla X_{t_{l-1}}^n}{\sigma(t_{l-1}, X_{t_{l-1}}^n) \nabla X_{t_k}^n} (B_{t_l}^n - B_{t_{l-1}}^n)$$

But $\hat{Z}_{tk}^n \neq Z_{tk}^n$, and $\mathbb{E} |\hat{Z}_{tk}^n - Z_{tk}^n|^2 \to 0$ only for $g''$ $\alpha$-Hölder, $f$ sufficiently smooth
Convergence Results

Theorem (C. Geiss, C.L., A. Luoto)

Let $b, \sigma$ and $f$ satisfy the above assumptions. Let $g''$ be a locally $\alpha$-Hölder continuous function and assume additionally that all first and second partial derivatives w.r.t. the variables $x, y, z$ of $b(t, x), \sigma(t, x)$ and $f(t, x, y, z)$ exist and are bounded Lipschitz functions w.r.t. these variables, uniformly in time. Then for all $t \in [0, T)$ and large enough $n$, we have

$$
\mathbb{E}_{0, x} |Y_t - Y_t^n|^2 \leq C\psi(x)h^{\frac{1}{2}}
$$

$$
\mathbb{E}_{0, x} |Z_t - Z_t^n|^2 \leq C\psi(x)h_{\frac{1}{2}}^{1 + \alpha},
$$

where $\psi(x) := K(1 + |x|^{p_0 + 1})$. 
Key results: regularity for fractional smoothness

Theorem (C. Geiss, S. Geiss, E. Gobet (2012))

If $g$ is $\alpha$-Hölder and $f$ Lipschitz, it holds for $0 \leq t < s < T$ and $x \in \mathbb{R}$,

\[ \| Y_s - Y_t \|_{L_p(P_{t,x})} \leq c_4 \psi(x) \left( \int_t^s (T - r)^{\alpha-1} dr \right)^{\frac{1}{2}}, \]

\[ \| Z_s - Z_t \|_{L_p(P_{t,x})} \leq c_5 \psi(x) \left( \int_t^s (T - r)^{\alpha-2} dr \right)^{\frac{1}{2}}. \]
Properties of the associated PDE

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) + \frac{\sigma^2(t, x)}{2} \frac{\partial^2 u}{\partial x^2}(t, x) + b(t, x) \frac{\partial u}{\partial x}(t, x) + f(t, x, u(t, x), (\sigma u_x)(t, x)) &= 0, \\
u(T, x) &= g(x), \quad x \in \mathbb{R},
\end{align*}
\]

\[t \in [0, T), x \in \mathbb{R},\]

Properties of \( u \) and \( u_x \) (Zhang (2005)) \( u_{xx} \) (C. Geiss, C.L, A. Luoto)

1. \( Y_t = u(t, X_t) \) and \( Z_{s,x}^t = u_x(s, X_s^t,x)\sigma(s, X_s^t,x) \)
   where \( u(t, x) = \mathbb{E}_{t,x} \left( g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r) dr \right) \),

2. \( u \) is continuous on \([0, T] \times \mathbb{R}, \) \( u_x \) and \( u_{xx} \) are continuous on \([0, T] \times \mathbb{R}, \)

3. \( |u(t, x)| \leq c_1 \psi(x), \quad |u_x(t, x)| \leq \frac{c_2 \psi(x)}{(T-t)^{1-\alpha}}, \quad |u_{xx}(t, x)| \leq \frac{c_3 \psi(x)}{(T-t)^{1-\alpha/2}}, \)

\[\partial_x^i u(t, x) = \mathbb{E}_{t,x} \left[ g(X_T) N_T^{t,i} + \int_t^T f(r, X_r, Y_r, Z_r) N_r^{t,i} dr \right], \quad \text{where } N_r^{t,i} \text{ denotes the Malliavin weight of the } i\text{th order.}\]
Conclusion

By using the Skorohod embedding to approximate the Brownian motion, we manage to prove,

- if $f$ is Lipschitz and $g$ is locally $\alpha$-Hölder and $Y_T = g(B_T)$ that
  - the rate of convergence $= \frac{\alpha}{4}$ for the $L^2$-error on $Y$ and $< \frac{\alpha}{4}$ for the $L^2$-error on $Z$,
- if $Y_T = g(X_T)$, $X$ a diffusion process with nice enough $b$ and $\sigma$, and
  - $g$ to be locally $C^{2,\alpha}$
  - $f$ has w.r.t. $x, y, z$ Lipschitz continuous second partial derivatives,
then
  - the rate of convergence $= \frac{1}{4}$ for the $L^2$-error on $Y$ and $= \frac{1}{4} \land \frac{\alpha}{2}$ for the $L^2$-error on $Z$