

Approximation rate of BSDEs using random walk

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Random walk approximation of the Brownian motion

Let $t_k := kh$, $k = 0, \dots, n$ be a regular grid of $[0, T]$, where $h = \frac{T}{n}$ and define

$$B_t^n := \sqrt{h} \sum_{k=1}^{\lceil t/h \rceil} \varepsilon_k, \quad (\varepsilon_k)_{k=1, \dots, n} \text{ i.i.d. Bernoulli r.v.:$$

$$\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}$$

- Donsker's Theorem:

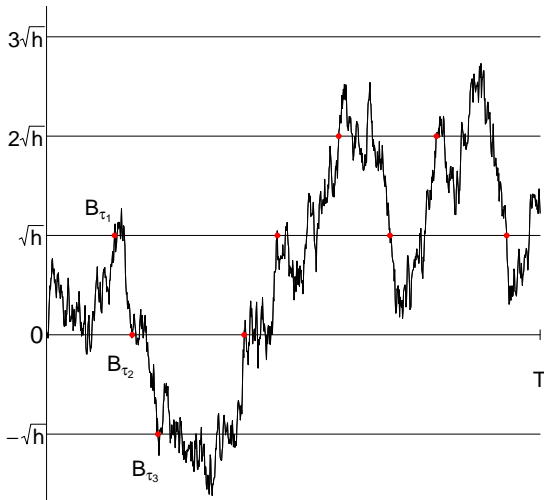
$$(B_t^n)_{t \in [0, T]} \rightarrow (B_t)_{t \in [0, T]} \quad \text{in distribution.}$$

- Wanted:

$$B_t^n \rightarrow B_t \quad \text{in } L_2 \quad \text{for all } t \in [0, T] \text{ with a convergence rate.}$$

Random walk constructed by Skorohod embedding

$$\tau_0 = 0, \quad \tau_k := \inf\{t > \tau_{k-1} : |B_t - B_{\tau_{k-1}}| = \sqrt{h}\}$$



Properties of $(B_{\tau_k})_k$ and $(\tau_k)_k$

- $(B_{\tau_k} - B_{\tau_{k-1}})_{k \geq 1} \stackrel{d}{=} (\sqrt{h}\varepsilon_k)_{k \geq 1}$
- $(\tau_k - \tau_{k-1})_k$ are i.i.d. $\sim \tau := \inf\{t > 0 : |B_t| = \sqrt{h}\}$
- It holds $\mathbb{E}(\tau_k) = t_k$, $\mathbb{E}|\tau_k - t_k|^2 = \frac{2}{3}t_k h$, $k = 1, \dots, n$.

$$\mathbb{E}|B_{\tau_k} - B_{t_k}|^2 = \mathbb{E}|\tau_k - t_k| \leq \sqrt{\frac{2}{3}t_k h}$$

- random walk constructed from B: $B_t^n := \sqrt{h} \sum_{k=1}^{\lceil t/h \rceil} (B_{\tau_k} - B_{\tau_{k-1}})$

Since $B_{t_k}^n = B_{\tau_k}$ we have for $t \in [t_k, t_{k+1})$

$$\begin{aligned} \mathbb{E}|B_t^n - B_t|^2 &\leq 2\mathbb{E}|B_{\tau_k} - B_{t_k}|^2 + 2\mathbb{E}|B_{t_k} - B_t|^2 \\ &\leq 2\sqrt{\frac{2}{3}t_k \frac{T}{n}} + 2\frac{T}{n} \leq 4\frac{T}{\sqrt{n}} \end{aligned}$$

Forward Backward Stochastic Differential Equations

$$X_t = x + \int_0^t b(r, X_r) dr + \int_0^t \sigma(r, X_r) dB_r, \quad 0 \leq t \leq T$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s$$

replace $(B_t)_{t \in [0, T]}$ by a random walk $(B_t^n)_{t \in [0, T]}$ what kind of convergence

$$((X_t^n)_{t \in [0, T]}, (Y_t^n)_{t \in [0, T]}, (Z_t^n)_{t \in [0, T]}) \rightarrow ((X_t)_{t \in [0, T]}, (Y_t)_{t \in [0, T]}, (Z_t)_{t \in [0, T]})?$$

Briand, Delyon and Mémin (2001)

If b, σ, f and g are Lipschitz and $(B_t^n)_{t \in [0, T]}$ such that

$\sup_{0 \leq t \leq T} |B_t^n - B_t| \rightarrow 0, \quad n \rightarrow \infty,$ **in probability**, then

$$\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 ds \rightarrow 0 \text{ when } n \rightarrow \infty \text{ in probability.}$$

Other results

- Toldo (2005) extends the previous result to BSDEs with random terminal time
- Numerical schemes : Ma, Protter, San Martin and Torres (2002), Peng, Xu (2008) and Mémin, Peng, Xu (2008) (Implicit and explicit schemes for BSDEs and RBSDEs), Martinez, San Martin and Torres (2011) (RBSDEs), Jańczyk (2008, 2009) (generalized RBSDEs with random terminal time)

⇒ convergence **in probability or weak convergence**: no rate

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Discretization

We consider the BSDE

$$Y_t = g(B_T) + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T$$

and its approximation

$$Y_{t_k}^n = g(B_T^n) + h \sum_{m=k}^{n-1} f(t_{m+1}, Y_{t_m}^n, Z_{t_m}^n) - \sum_{m=k}^{n-1} Z_{t_m}^n \underbrace{(B_{t_{m+1}}^n - B_{t_m}^n)}_{\substack{\text{law} \\ = \sqrt{h}\varepsilon_{m+1}}}$$

For n large enough: $\exists!$ $(\mathcal{F}_{\tau_k})_k$ -adapted solution $(Y_{t_k}^n, Z_{t_k}^n)_{k=0}^{n-1}$.

The corresponding scheme:

$$Y_{t_n}^n := g\left(\sum_{k=1}^n \sqrt{h}\varepsilon_k\right)$$

$$Z_{t_{k-1}}^n := \frac{\mathbb{E}_{\tau_{k-1}}[Y_{t_k}^n \varepsilon_k]}{\sqrt{h}}, \quad k = 1, \dots, n$$

$$Y_{t_{k-1}}^n := \mathbb{E}_{\tau_{k-1}}[Y_{t_k}^n] + f(t_k, Y_{t_{k-1}}^n, Z_{t_{k-1}}^n)h$$

Discretization : approximation of Z

From Zhang's paper (2005), we have

$$Z_t = \mathbb{E}_t \left(g(B_T) \frac{B_T - B_t}{T - t} \right) + \mathbb{E}_t \left(\int_t^T f(s, Y_s, Z_s) \frac{B_s - B_t}{s - t} ds \right),$$

Lemma

For all $k = 0, 1, \dots, n - 1$, we have

$$Z_{t_k}^n = \mathbb{E}_{\tau_k} \left(g(B_T^n) \frac{B_{t_n}^n - B_{t_k}^n}{t_n - t_k} \right) + h \mathbb{E}_{\tau_k} \left(\sum_{m=k+1}^{n-1} f(t_{m+1}, Y_{t_m}^n, Z_{t_m}^n) \frac{B_{t_m}^n - B_{t_k}^n}{t_m - t_k} \right).$$

Convergence results

Assume: $\forall (t, y, z), (t', y', z') \in [0, T] \times \mathbf{R}^2$,

$$|f(t, y, z) - f(t', y', z')| \leq L_f(\sqrt{|t - t'|} + |y - y'| + |z - z'|),$$

$$\forall (x, x') \in \mathbf{R}^2, \quad |g(x) - g(x')| \leq C_g(1 + |x|^{p_0} + |x'|^{p_0})|x - x'|^\alpha.$$

Theorem (C.G., Labart, Luoto)

$$\sup_{0 \leq t < T} \mathbb{E}|Y_t - Y_t^n|^2 \leq C_1 h^{\frac{\alpha}{2}},$$

$$\mathbb{E} \int_0^T |Z_t - Z_t^n|^2 dt \leq C_2 h^\beta \quad \text{for } \beta \in (0, \frac{\alpha}{2})$$

where C_1 depends on L_f, T, C_g, p_0 and α and C_2 additionally on β .

The reason for β :

$$\mathbb{E}|Z_t - Z_t^n|^2 \leq C_0 \frac{h^{\frac{\alpha}{2}}}{T - t_k} + C_1 \frac{h^{\frac{\alpha}{2}}}{(T - t)^{1 - \frac{\alpha}{2}}} \mathbf{1}_{t \neq t_k} \quad \text{for } t \in [t_k, t_{k+1}), \quad k = 0, \dots, n-1,$$

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Example 1

$$dY_t = -(Y_t + Z_t)dt + Z_t dB_t,$$
$$Y_T = B_T^2$$

The explicit solution is given by

$$Y_t = e^{T-t}((B_t + T - t)^2 + T - t), \quad Z_t = 2e^{T-t}(B_t + T - t)$$

We compute by Monte Carlo $\mathbb{E}(|Y_t - Y_t^n|^2)$ and $\mathbb{E}(|Z_t - Z_t^n|^2)$ for different values of n .

g is a locally Lipschitz function : the convergence should go faster than $\frac{1}{\sqrt{n}}$.

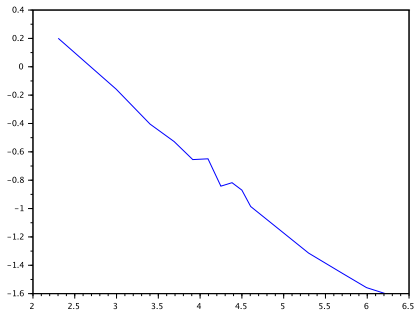


Figure: $\log(\text{Error on } Y)$ for different values of $\log(n)$

We get a slope of -0.46

Example 1 : error on Z

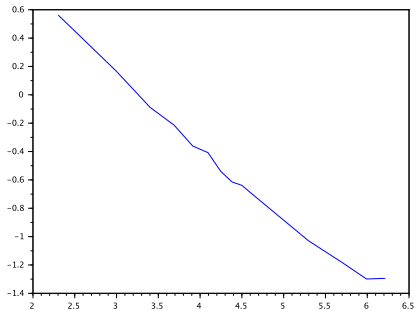


Figure: $\log(\text{Error on } Z)$ for different values of $\log(n)$

We get a slope of -0.48

Example 2

$$dY_t = -(Y_t + Z_t)dt + Z_t dB_t,$$
$$Y_T = \sqrt{|B_T|}$$

The solution is given by

$$Y_t = e^{\frac{T-t}{2}} \tilde{E}(\sqrt{|\tilde{B}_{T-t} + B_t|} e^{\tilde{B}_{T-t}})$$

We compute by Monte Carlo $\mathbb{E}(|Y_t - Y_t^n|^2)$ and $\mathbb{E}(|Z_t - Z_t^n|^2)$ for different values of n .

g is a $\frac{1}{2}$ -Hölder function : the convergence should go faster $\frac{1}{n^4}$.

Example 2 : error on Y

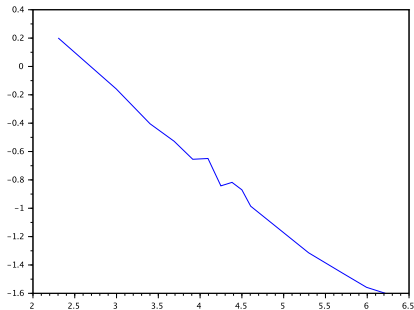


Figure: $\log(\text{Error on } Y)$ for different values of $\log(n)$

We get a slope of $-0.56!!!!$

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The general case

$$X_t = x + \int_0^t b(r, X_r) dr + \int_0^t \sigma(r, X_r) dB_r, \quad 0 \leq t \leq T$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s$$

- $b, \sigma \in C_b^{0,2}$ and $\frac{1}{2}$ -Hölder in time, unif. in space
- the first and second derivatives w.r.t. the space variable are assumed to be γ -Hölder continuous (for some $\gamma \in]0, 1]$, w.r.t. the parabolic metric $d((x, t), (x', t')) = (|x - x'|^2 + |t - t'|)^{\frac{1}{2}}$) on all compact subsets of $[0, T] \times \mathbf{R}$.
- $\sigma(t, x) \geq \delta > 0$

Numerical Scheme for X and Y

$$X_0^n = x,$$

$$X_{t_k}^n = x + \sum_{j=1}^k b(t_j, X_{t_{j-1}}^n)h + \sqrt{h} \sum_{j=1}^k \sigma(t_j, X_{t_{j-1}}^n) \varepsilon_j$$

$$Y_{t_k}^n = g(X_T^n) + h \sum_{m=k}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) - \sqrt{h} \sum_{m=k}^{n-1} Z_{t_m}^n \varepsilon_{m+1}.$$

and

$$Y_{t_k}^n = \mathbb{E}_{\tau_k} \left(g(X_{t_n}^n) + h \sum_{m=k}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \right)$$

$$Z_{t_k}^n = \frac{\mathbb{E}_{\tau_k} g(X_{t_n}^n) \varepsilon_{k+1}}{\sqrt{h}} + \mathbb{E}_{\tau_k} \left(\sqrt{h} \sum_{m=k+1}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \varepsilon_{k+1} \right)$$

Approximation of Z

$$Z_t = \mathbb{E}_t \left(g(X_T) N_T^t + \int_t^T f(r, X_r, Y_r, Z_r) N_r^t dr \right) \sigma(t, X_t), \quad \text{Zhang (2005)}$$

where $N_v^t := \frac{1}{v-t} \int_t^v \frac{\nabla X_s}{\sigma(s, X_s) \nabla X_t} dB_s$ is the Malliavin weight of first order

$$\widehat{Z}_{t_k}^n := \mathbb{E}_{\tau_k} \left(g(X_T^n) N_{\tau_n}^{n, \tau_k} + h \sum_{m=k+1}^{n-1} f(t_{m+1}, X_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) N_{\tau_m}^{n, \tau_k} \right),$$

$$N_{\tau_m}^{n, \tau_k} = \frac{1}{t_m - t_k} \sum_{l=k+1}^m \frac{\nabla X_{t_{l-1}}^n}{\sigma(t_{l-1}, X_{t_{l-1}}^n) \nabla X_{t_k}^n} (B_{t_l}^n - B_{t_{l-1}}^n)$$

But $\widehat{Z}_{t_k}^n \neq Z_{t_k}^n$, and $\mathbb{E} |\widehat{Z}_{t_k}^n - Z_{t_k}^n|^2 \rightarrow 0$ only for g'' α -Hölder, f sufficiently smooth

Convergence Results

Theorem (C. Geiss, C.L., A. Luoto)

Let b, σ and f satisfy the above assumptions. Let g'' be a **locally α -Hölder** continuous function and **assume additionally that all first and second partial derivatives w.r.t. the variables x, y, z of $b(t, x), \sigma(t, x)$ and $f(t, x, y, z)$ exist and are bounded Lipschitz functions w.r.t. these variables, uniformly in time.** Then for all $t \in [0, T)$ and large enough n , we have

$$\begin{aligned}\mathbb{E}_{0,x} |Y_t - Y_t^n|^2 &\leq C\psi(x)h^{\frac{1}{2}} \\ \mathbb{E}_{0,x} |Z_t - Z_t^n|^2 &\leq C\psi(x)h^{\frac{1}{2} \wedge \alpha},\end{aligned}$$

where $\psi(x) := K(1 + |x|^{\rho_0+1})$.

Key results : regularity for fractional smoothness

Theorem (C. Geiss, S. Geiss, E. Gobet (2012))

If g is α -Hölder and f Lipschitz, it holds for $0 \leq t < s < T$ and $x \in \mathbf{R}$,

$$\|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \leq c_4 \psi(x) \left(\int_t^s (T-r)^{\alpha-1} dr \right)^{\frac{1}{2}},$$

$$\|Z_s - Z_t\|_{L_p(\mathbb{P}_{t,x})} \leq c_5 \psi(x) \left(\int_t^s (T-r)^{\alpha-2} dr \right)^{\frac{1}{2}}.$$

Properties of the associated PDE

$$\left\{ \begin{array}{l} u_t(t, x) + \frac{\sigma^2(t, x)}{2} u_{xx}(t, x) + b(t, x) u_x(t, x) + f(t, x, u(t, x), (\sigma u_x)(t, x)) = 0, \\ u(T, x) = g(x), \quad x \in \mathbf{R} \end{array} \right. \quad t \in [0, T), x \in \mathbf{R},$$

Properties of u and u_x (Zhang (2005)) u_{xx} (C. Geiss, C.L. A. Luoto)

1. $Y_t = u(t, X_t)$ and $Z_s^{t,x} = u_x(s, X_s^{t,x}) \sigma(s, X_s^{t,x})$

where

$$u(t, x) = \mathbb{E}_{t,x} \left(g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r) dr \right),$$

2. u is continuous on $[0, T] \times \mathbf{R}$, u_x and u_{xx} are continuous on $[0, T) \times \mathbf{R}$,

3. $|u(t, x)| \leq c_1 \psi(x)$, $|u_x(t, x)| \leq \frac{c_2 \psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}$, $|u_{xx}(t, x)| \leq \frac{c_3 \psi(x)}{(T-t)^{1-\frac{\alpha}{2}}}$,

$$\partial_x^i u(t, x) = \mathbb{E}_{t,x} \left[g(X_T) N_T^{t,i} + \int_t^T f(r, X_r, Y_r, Z_r) N_r^{t,i} dr \right], \text{ where } N_r^{t,i} \text{ denotes}$$

the Malliavin weight of the i th order

Conclusion

By using the Skorohod embedding to approximate the Brownian motion, we manage to prove,

- if f is Lipschitz and g is locally α -Hölder and $Y_T = g(B_T)$ that
 - the rate of convergence = $\frac{\alpha}{4}$ for the L^2 -error on Y and $< \frac{\alpha}{4}$ for the L^2 -error on Z ,
- if $Y_T = g(X_T)$, X a diffusion process with nice enough b and σ , and
 - g to be locally $C^{2,\alpha}$
 - f has w.r.t. x, y, z Lipschitz continuous second partial derivatives,then
 - the rate of convergence = $\frac{1}{4}$ for the L^2 -error on Y and = $\frac{1}{4} \wedge \frac{\alpha}{2}$ for the L^2 -error on Z