The inverse of the dispersion depends logarithmically on the dimension

(joint work with J. Vybíral)

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The Dispersion

In the following, we let $\mathcal{P}_n$ be a **point set** in $[0,1]^d$ with $\#\mathcal{P}_n = n$.

We define the **dispersion** of $\mathcal{P}_n$ by

$$\mathrm{disp}(\mathcal{P}_n) := \sup_{B\colon B\cap\mathcal{P}_n=\varnothing} |B|,$$

where the sup is over all (axis-parallel) boxes $B = I_1 \times \cdots \times I_d$.

That is, we're looking for the volume of the **"largest empty box"**.

The Minimal Dispersion

In particular, we are interested in the *best possible* dispersion.

For this, we introduce the n-**th minimal dispersion**

$$\mathrm{disp}(n,d) \;:=\; \inf_{\mathcal{P}_n} \mathrm{disp}(\mathcal{P}_n)$$

as well as its **inverse**

$$N(\varepsilon,d) \;:=\; \min\{n \colon \mathrm{disp}(n,d) \le \varepsilon\}.$$

Applications

Although quite unstudied, there are already interesting applications:

- interesting geometric quantity (Rote, Hlawka, Tichy)
- optimization (Niederreiter, L'Ecuyer)
- Approximation of rank-1 tensors (Bachmayr, Dahmen, DeVore, Grasedyk; Novak, Rudolf; Krieg, Rudolf)
- Marcinkiewicz-type discretization, i.e., approximation of L_p-norms of certain trig. polynomials using function values (Temlyakov)

However, we do not have a precise equivalence to a numerical problem. (As far as I know...)

Discrepancy

A more popular quantity is the **discrepancy** of a point set $\mathcal{P}_n$, which is defined by

$$D(\mathcal{P}_n) := \sup_B \left| \frac{\#(B \cap \mathcal{P}_n)}{n} - |B| \right|, \qquad D(n,d) = \inf_{\mathcal{P}_n} D(\mathcal{P}_n).$$

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We know that

$$D(\mathcal{P}_n) \approx \sup_{f \colon \|f'\|_1 \le 1} \left| \frac{1}{n} \sum_{x \in \mathcal{P}_n} f(x) - \int_{[0,1]^d} f(y) \, \mathrm{d}y \right|$$

and this shows the connection of such geometric problems to various other field of mathematics.

Discrepancy: Known results

There are plenty of results on $D(n,d)$, e.g.,

- $D(n,2) \asymp \dfrac{\log(n)}{n}$ (Schmidt)
- $\dfrac{\log^{\frac{d-1}{2}}(n)}{n} \lesssim_d D(n,d) \lesssim_d \dfrac{\log^{d-1}(n)}{n}$ (Roth; Halton)
- $\dfrac{\log^{\frac{d-1}{2}+\frac{c}{d^2}}(n)}{n} \lesssim_d D(n,d)$ (Bilyk, Lacey, Vagharshakyan)

The order (in n) of $D(n,d)$ is still unknown!

Conjecture: $D(n,d) \asymp \dfrac{\log^{d-1}(n)}{n}$ or $D(n,d) \asymp \dfrac{\log^{\frac{d}{2}}(n)}{n}$

Discrepancy: Known results II

The previous results do not lead to any nontrivial and **explicit-in-d** bound on

$$n(\varepsilon, d) := \min\{n \colon D(n,d) \le \varepsilon\}.$$

But it was proven subsequently that

$$\frac{d}{\varepsilon} \lesssim n(\varepsilon, d) \lesssim \frac{d}{\varepsilon^2} \qquad (\textit{Hinrichs}; \textit{HNWW})$$

or $d/n \lesssim D(n,d) \lesssim \sqrt{d/n}$.

That is, $n(\varepsilon, d)$ is **linear** in d, but we don't know the order in ε!

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Conjecture: We do not even have one...

(Some conjecture that $\liminf_{\varepsilon\to 0} \varepsilon^c n(\varepsilon, d) \ge (1+\gamma)^d$ for $c < 2$ and $\gamma > 0$.)

Dispersion: Known results

It is rather easy to prove

$$\mathrm{disp}(n, d) \asymp_d \frac{1}{n} \qquad \left(\text{or} \quad N(\varepsilon, d) \asymp_d \frac{1}{\varepsilon}\right).$$

Hence, we know the optimal order in n.

Regarding the d-dependence, it was proven recently that

- $N(\varepsilon, d) \gtrsim \dfrac{\log(d)}{\varepsilon}$ (Aistleitner, Hinrichs, Rudolf)
- $N(\varepsilon, d) \lesssim \dfrac{d \log(1/\varepsilon)}{\varepsilon}$ (Rudolf)
- $N(\varepsilon, d) \lesssim \varepsilon^{-\varepsilon^{-2}} \log(d)$ (Sosnovec)

New Result

Known:

$$\frac{\log(d)}{\varepsilon} \lesssim N(\varepsilon, d) \lesssim \varepsilon^{-\varepsilon^{-2}} \log(d)$$

Theorem (U, Vybíral, '18)

For $d \geq 2$ and $\varepsilon < \frac{1}{4}$, we have

$$N(\varepsilon, d) \leq 2^7 \log_2(d) \left(\frac{\log_2(1/\varepsilon)}{\varepsilon} \right)^2 .$$

Actually, we show that random points chosen in $\left[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right]^d$ satisfy this bound with positive probability.

Explicit constructions

The only explicit (i.e. polynomial-time) constructions known so far are

- digital nets: $\mathrm{disp}(\mathcal{P}_n) \leq \frac{2^{7d+2}}{n}$ (Larcher)
- sparse grids: $\mathrm{disp}(\mathcal{P}_n) \leq n^{-\frac{1}{\log_2(d)}}$ (Krieg)

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Work in progress: For some $c > 0$, there are explicit $\mathcal{P}_n$ with
(still with Jan)

$$\mathrm{disp}(\mathcal{P}_n) \lesssim \left(\frac{\log(d)}{n}\right)^c.$$

(Based on deep results from the theory of self-correcting codes.)

Thank you!