Sensitivity and Robustness of Financial Models

Giray Ökten
Joint work with David Mandel
Florida State University
Sobol’ global sensitivity analysis: Notation

- Model: \( f = f(x_1, \ldots, x_s) \)
- Scale the input variables to \((0, 1)\)
- Index set \( D = \{1, 2, \ldots, s\} \)
- \( u \subseteq D \), \(-u\) is the complement of \( u \)
- \( f_u(x^u) \) is a function that only depends on \( x^u \)
Anova Decomposition of Functions

\[ f(x) = \sum_{u \subseteq D} f_u(x^u) \]

- Component functions \( f_u \) are constructed recursively:

\[ f_u(x) = \int f(x) \, dx^u - \sum_{v \subset u} f_v(x^v) \]

- Decomposition is orthogonal, and, as a consequence

\[ \sigma^2 = \sum_{u \subseteq D} \sigma^2_u \]

where

\[ \sigma^2 = \int f^2(x) \, dx - \left( \int f(x) \, dx \right)^2 \]

\[ \sigma^2_u = \int f_u^2(x) \, dx^u \]
Sensitivity indices

- Sobol' calls $\sigma_u^2 / \sigma^2$ the global sensitivity index.
- Different ways to measure importance of a set of variables:

$$S_u = \frac{1}{\sigma^2} \sum_{v \subseteq u} \sigma_v^2 = \frac{\tau_u}{\sigma^2}$$

$$\overline{S}_u = \frac{1}{\sigma^2} \sum_{v \cap u \neq \emptyset} \sigma_v^2 = \frac{\overline{\tau}_u}{\sigma^2}$$
Use the sensitivity indices to identify the “unimportant” parameters of the model, and thus reduce the dimension of the model:

- Compute $\bar{S}_{\{i\}} = \frac{1}{\sigma^2} \sum_{\nu \cap \{i\} \neq \emptyset} \sigma^2_{\nu}$
- Freeze variable $x_i$ at its constant value if $\bar{S}_{\{i\}}$ is small
Example: Rothermel’s Fire Spread Model

- Output:
  - rate of spread (ros)
  - direction of max spread (sdr)
  - effective wind speed (efw)
  - reaction intensity (ri)
Example: Rothermel’s Fire Spread Model

- **Output:**
  - rate of spread (ros)
  - direction of max spread (sdr)
  - effective wind speed (efw)
  - reaction intensity (ri)

- **Input:** There are a lot of them...
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>fuel bed depth</td>
<td>$d$</td>
<td>1.83</td>
<td>m</td>
</tr>
<tr>
<td>low heat content</td>
<td>heat</td>
<td>18622.0</td>
<td>kJ/kg</td>
</tr>
<tr>
<td>1-h fuel moisture</td>
<td>$m_{d1}$</td>
<td>8.0</td>
<td>%</td>
</tr>
<tr>
<td>10-h fuel moisture</td>
<td>$m_{d2}$</td>
<td>8.0</td>
<td>%</td>
</tr>
<tr>
<td>100-h fuel moisture</td>
<td>$m_{d3}$</td>
<td>8.0</td>
<td>%</td>
</tr>
<tr>
<td>live herbaceous fuel moisture</td>
<td>$m_{lh}$</td>
<td>150.0</td>
<td>%</td>
</tr>
<tr>
<td>live woody fuel moisture</td>
<td>$m_{lw}$</td>
<td>150.0</td>
<td>%</td>
</tr>
<tr>
<td>moisture of extinction</td>
<td>$m_{x}$</td>
<td>20</td>
<td>%</td>
</tr>
<tr>
<td>particle density</td>
<td>$\rho_p$</td>
<td>512.5</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>effective mineral content</td>
<td>$s_e$</td>
<td>1.0</td>
<td>%</td>
</tr>
<tr>
<td>slope</td>
<td>$s_{lp}$</td>
<td>14.04</td>
<td>◦</td>
</tr>
<tr>
<td>total mineral content</td>
<td>$s_t$</td>
<td>5.55</td>
<td>%</td>
</tr>
<tr>
<td>1-h surface area/vol ratio</td>
<td>$sv_{d1}$</td>
<td>6562.0</td>
<td>m$^2$/m$^3$</td>
</tr>
<tr>
<td>10-h surface area/vol ratio</td>
<td>$sv_{d2}$</td>
<td>358.0</td>
<td>m$^2$/m$^3$</td>
</tr>
<tr>
<td>100-h surface area/vol ratio</td>
<td>$sv_{d3}$</td>
<td>98.0</td>
<td>m$^2$/m$^3$</td>
</tr>
<tr>
<td>live herb surface area/vol ratio</td>
<td>$sv_{lh}$</td>
<td>4921.0</td>
<td>m$^2$/m$^3$</td>
</tr>
<tr>
<td>live woody surface area/vol ratio</td>
<td>$sv_{lw}$</td>
<td>4921.0</td>
<td>m$^2$/m$^3$</td>
</tr>
<tr>
<td>direction of wind vector (from upslope)</td>
<td>$\theta$</td>
<td>45</td>
<td>◦</td>
</tr>
<tr>
<td>1-h fuel load</td>
<td>$w_0{d1}$</td>
<td>1.12</td>
<td>kg/m$^2$</td>
</tr>
<tr>
<td>10-h fuel load</td>
<td>$w_0{d2}$</td>
<td>0.90</td>
<td>kg/m$^2$</td>
</tr>
<tr>
<td>100-h fuel load</td>
<td>$w_0{d3}$</td>
<td>0.45</td>
<td>kg/m$^2$</td>
</tr>
<tr>
<td>live herbaceous fuel load</td>
<td>$w_0{lh}$</td>
<td>0</td>
<td>kg/m$^2$</td>
</tr>
<tr>
<td>live woody fuel load</td>
<td>$w_0{lw}$</td>
<td>1.12</td>
<td>kg/m$^2$</td>
</tr>
<tr>
<td>midflame wind speed</td>
<td>$w_{sp}$</td>
<td>2.3</td>
<td>m/s</td>
</tr>
</tbody>
</table>
Figure 1: Sobol sensitivity analysis, left-right, up-down: ros, sdr, efw, ri
Rothermel’s Fire Spread Model

- Most input parameters had a normal distribution - measurement error - and a few had empirical distributions obtained from field data
- Only 7 input variables out of 24 had a significant effect on the model variance
- Used RQMC and variance reduction techniques for uncertainty quantification for the reduced model
Application: Interest rate models

Vasicek: \[ dr(t) = a(b - r(t))dt + \sigma dW(t) \]

CIR: \[ dr(t) = a(b - r(t))dt + \sigma \sqrt{r(t)}dW(t) \]

Calibration:

- Use one year of interest rate data (yields on one-year T-bills)
- Use MLE to estimate \( a, b, \sigma \)
## Numerical results

<table>
<thead>
<tr>
<th>Year</th>
<th>Parameter</th>
<th>Vasicek</th>
<th>CIR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>a</td>
<td>4.05(2.62)</td>
<td>3.36 (0.96)</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>7.61(0.46)</td>
<td>2.21 (0.54)</td>
</tr>
<tr>
<td></td>
<td>σ</td>
<td>1.06(4.53)</td>
<td>1.07 (0.47)</td>
</tr>
<tr>
<td>1987</td>
<td>a</td>
<td>4.73(2.54)</td>
<td>2.72 (0.92)</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>7.02(0.33)</td>
<td>1.49 (0.60)</td>
</tr>
<tr>
<td></td>
<td>σ</td>
<td>1.05(3.65)</td>
<td>1.08 (0.37)</td>
</tr>
<tr>
<td>2006</td>
<td>a</td>
<td>6.26(2.14)</td>
<td>4.32 (1.07)</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>2.72(0.08)</td>
<td>0.90 (0.12)</td>
</tr>
<tr>
<td></td>
<td>σ</td>
<td>1.02(0.94)</td>
<td>1.02 (0.21)</td>
</tr>
</tbody>
</table>

Parameter estimates and standard errors for three calibrations
<table>
<thead>
<tr>
<th>Year</th>
<th>Sensitivity</th>
<th>Vasicek</th>
<th>CIR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S_a$</td>
<td>0.57</td>
<td>0.25</td>
</tr>
<tr>
<td>1974</td>
<td>$S_b$</td>
<td>0.43</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>$S_\sigma$</td>
<td>0.004</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>$S_a$</td>
<td>0.38</td>
<td>0.23</td>
</tr>
<tr>
<td>1987</td>
<td>$S_b$</td>
<td>0.63</td>
<td>0.78</td>
</tr>
<tr>
<td></td>
<td>$S_\sigma$</td>
<td>0.002</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>$S_a$</td>
<td>0.0005</td>
<td>0.007</td>
</tr>
<tr>
<td>2006</td>
<td>$S_b$</td>
<td>0.99</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>$S_\sigma$</td>
<td>0.0002</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Sensitivity indices for three calibrations
What’s important, what’s not: an interesting behavior

- For each year (1962-2015) estimate the model parameters using maximum likelihood method
- The sampling distribution of the parameters is asymptotically normal
- Compute the Sobol’ upper sensitivity index for each parameter, using its sampling distribution
Upper Sobol’ sensitivity indices for Vasicek and CIR models across years

Observe: What is important seems to change from year to year!
• Three models for daily average temperature: [Alaton 2002], [Benth 2007] and [Brody 2002]
• Number of parameters: 17 in Alaton, 14 in Benth, 18 in Brody
• Parameters estimated for twenty-five locations in the US
Weather Derivatives

- Heating degree-day (HDD) on day $t \in \mathbb{N}$:

$$HDD(t) := \max\{65 - T(t), 0\}$$

- Total HDDs in contract period $[t_0, t_N]$:

$$H(t_N) := \sum_{n=0}^{N} HDD(t_n)$$

- European call option on total HDDs in $[0, T]$:

$$C = e^{-rT} E^P \left( \max\{H(t_N) - K, 0\} \right)$$

where $K$ is the strike
Alaton and Benth models assume the average daily temperature follows

\[ dT(t) = ds(t) + a(s(t) - T(t)) dt + \sigma(t) dW(t), \]

whereas Brody assumes

\[ dT(t) = ds(t) + a(s(t) - T(t)) dt + \sigma(t) dW^H(t), \]

where

\[ s(t) = A + Bt + C \sin(\omega t) + D \cos(\omega t) \]

and \( \omega = \frac{2\pi}{365} \).

- Alaton and Brody assume \( \sigma(t) \) piecewise constant in each month:
  \[ \sigma(t) \in \{\sigma_{Jan}, \sigma_{Feb}, \ldots, \sigma_{Dec}\} \]

- Benth assumes a truncated Fourier series:
  \[ \sigma(t) = c_0 + \sum_{i=1}^{4} c_i \sin(i\omega t) + \sum_{j=1}^{4} d_j \cos(j\omega t) \]

- \( H \in (0, 1) \) is the Hurst parameter in Brody
What’s important, what’s not

- For each location, we use data from that location to estimate the model parameters using maximum likelihood method.
- The sampling distribution of the parameters is asymptotically normal.
- We compute the Sobol’ upper sensitivity index for each parameter, using its sampling distribution. Here is how the index changes across locations, for the Benth model.
What’s important, what’s not

Sobol’ Indices Across Applications: Benth Model

Location

EWR LAX DTW MCO BOS

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7

S

Sobol’ Indices Across Applications: Benth Model

A
B
D
a

Location

EWR LAX DTW MCO BOS
Observations and Remedies

Observations:

- Sensitivity indices are not fixed, but random
- Ranking of parameters in terms of sensitivity may not be stable for a given model

Remedy:

- Randomized Sobol' indices and Anova decomposition
- Robustness of a model in terms of the stability of the sensitivity of its parameters
Observations and Remedies

Observations:

- Sensitivity indices are not fixed, but random
- Ranking of parameters in terms of sensitivity may not be stable for a given model

Remedy:

- Randomized Sobol’ indices and Anova decomposition
- Robustness of a model in terms of the stability of the sensitivity of its parameters
Randomized Sobol’ indices

Before:

\[
S_u = \frac{1}{\sigma^2} \sum_{v \subseteq u} \sigma_v^2
\]

\[
\overline{S}_u = \frac{1}{\sigma^2} \sum_{v \cap u \neq \emptyset} \sigma_v^2
\]

After:

\[
S_u^Y = \frac{1}{(\sigma^Y)^2} \sum_{v \subseteq u} (\sigma_v^Y)^2
\]

\[
\overline{S}_u^Y = \frac{1}{(\sigma^Y)^2} \sum_{v \cap u \neq \emptyset} (\sigma_v^Y)^2
\]
Randomized Anova decomposition

Before:

\[ f(x) = \sum_{u \subseteq D} f_u(x_u) \]

\[ \int_{[0,1]} f_u(x_i, x_u \setminus \{i\}) \, dx_i = 0 \]

After:

\[ f^Y(X) = \sum_{u \subseteq D} f^Y_u(X_u) \]

\[ \int_{R} f^Y_u(x_i, X_u \setminus \{i\}) \Lambda(dx_i \mid Y) = 0 \]
“Robustness is the insensitivity to small deviations from the assumptions”
"Robustness is the insensitivity to small deviations from the assumptions"

- A model with lower average variance across applications is more robust than a model with higher variance.
“Robustness is the insensitivity to small deviations from the assumptions”

- A model with lower average variance across applications is more robust than a model with higher variance.
- A model with a sensitivity pattern (ordering of Sobol’ sensitivity indices) that stays fixed with a higher probability across applications is more robust than one with a lower probability.
Notation

If $d = 3$ then

$$D! = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$$

If $\ell = (2,1,3)$ then

$$P(\bar{S}_\ell) = P(\bar{S}_2 < \bar{S}_1 < \bar{S}_3)$$
**Definition**

Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) and let \( D! \) denote the set of all permutations of the index set \( D = \{1, \ldots, d\} \).

For each \( \ell \in D! \), let \( \overline{S}_\ell \) denote the event \((\overline{S}_{\ell_1} < \overline{S}_{\ell_2} < \ldots, \overline{S}_{\ell_d})\). The robustness of the mathematical model \( f \) is

\[
R = \max_{\ell \in D!} \frac{P(\overline{S}_\ell^Y)}{\text{CoV}}
\]

where \( \text{CoV} = \frac{\sqrt{E(\text{Var}(f(X)|Y))}}{E(f(X))} \).
**Definition**

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and let $\tilde{D}!$ denote the set of all permutations of the potentially important index set $\tilde{D} = \{i_1, \ldots, i_p\}$, $p \leq d$.

For each $\ell \in D!$, let $\bar{S}_\ell$ denote the event $(\bar{S}_{\ell_1} < \bar{S}_{\ell_2} < \ldots, \bar{S}_{\ell_d})$. The robustness of the mathematical model $f$ is

$$R = \max_{\ell \in \tilde{D}!} \frac{P(\bar{S}_\ell^Y)}{\text{CoV}}$$

where $\text{CoV} = \sqrt{\frac{E(\text{Var}(f(X)|Y))}{E(f(X))}}$. 
## Robustness of interest rate models

<table>
<thead>
<tr>
<th>Event</th>
<th>Prob (Vasicek)</th>
<th>Prob (CIR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_a &lt; S_b &lt; S_\sigma$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$S_a &lt; S_\sigma &lt; S_b$</td>
<td>0.001</td>
<td>0.0</td>
</tr>
<tr>
<td>$S_b &lt; S_a &lt; S_\sigma$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$S_b &lt; S_\sigma &lt; S_a$</td>
<td>0.003</td>
<td>0.0</td>
</tr>
<tr>
<td>$S_\sigma &lt; S_a &lt; S_b$</td>
<td>0.53</td>
<td>0.69</td>
</tr>
<tr>
<td>$S_\sigma &lt; S_b &lt; S_a$</td>
<td>0.47</td>
<td>0.31</td>
</tr>
</tbody>
</table>

Parameter estimates and standard errors for three calibrations
Robustness of interest rate models

Results:

\[ R_{\text{Vasicek}} = \frac{0.53}{\sqrt{0.00226/0.97}} = 10.86 \]

\[ R_{\text{CIR}} = \frac{0.69}{\sqrt{0.00183/0.97}} = 15.7 \]

Therefore the CIR model is approximately \( 15.7/10.86 = 1.45 \) times more robust than the Vasicek model.
Varying Sobol’ Indices & Potentially Important Parameters

Sobol’ Indices Across Applications: Benth Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Alaton</th>
<th>Benth</th>
<th>Brody</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.88</td>
<td>0.92</td>
<td>0.80</td>
</tr>
<tr>
<td>B</td>
<td>0.98</td>
<td>0.99</td>
<td>0.94</td>
</tr>
<tr>
<td>C</td>
<td>&lt; 0.05</td>
<td>&lt; 0.05</td>
<td>&lt; 0.05</td>
</tr>
<tr>
<td>D</td>
<td>0.06</td>
<td>0.12</td>
<td>0.15</td>
</tr>
<tr>
<td>a</td>
<td>0.71</td>
<td>0.73</td>
<td>0.77</td>
</tr>
<tr>
<td>H</td>
<td>-</td>
<td>-</td>
<td>0.16</td>
</tr>
<tr>
<td>σ₁</td>
<td>0.48</td>
<td>-</td>
<td>0.56</td>
</tr>
<tr>
<td>σ₂, . . . , σ₁₂</td>
<td>&lt; 0.05</td>
<td>-</td>
<td>&lt; 0.05</td>
</tr>
<tr>
<td>c₀, . . . , c₄</td>
<td>-</td>
<td>&lt; 0.05</td>
<td>-</td>
</tr>
<tr>
<td>d₁</td>
<td>-</td>
<td>0.07</td>
<td>-</td>
</tr>
<tr>
<td>d₂, d₃, d₄</td>
<td>-</td>
<td>&lt; 0.05</td>
<td>-</td>
</tr>
</tbody>
</table>

Estimated $P(\bar{S}_i > 0.1)$
Robustness of the temperature models

Order Probabilities - Benth

\[ R_{Alaton} = \frac{0.18}{\sqrt{73.67/64.57}} = 1.32 \]

\[ R_{Benth} = \frac{0.26}{\sqrt{73.86/64.72}} = 1.92 \]

\[ R_{Brody} = \frac{0.22}{\sqrt{74.69/66.44}} = 1.67 \]

\[ R_{Alaton} < R_{Brody} < R_{Benth} \]
Questions?

Contact: okten@math.fsu.edu