Uniform estimates for particle filters

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Synthesis joint works with : A.N. Bishop, K. Kamatani, A. Kurtzmann, S. Pathiraja, B. Rémillard, J. Tugaut

Filtering and smoothing problems

$$\begin{cases} X_t & \text{Signal} = \text{Markov process on } \mathbb{R}^{r_1} \\ Y_t & \text{Observation (partial+noisy) on } \mathbb{R}^{r_2} \end{cases}$$

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Pb: Find/compute/sample/... sequentially

$$\boldsymbol{\eta}_t = \text{Law}(\boldsymbol{X_t} \mid \boldsymbol{Y_t}) \quad \text{or the marginal} \quad \eta_t = \text{Law}(X_t \mid \boldsymbol{Y_t})$$

with the historical processes

$$\boldsymbol{X_t} := (X_s)_{s \leq t} \quad \text{and} \quad \boldsymbol{Y_t} := (Y_s)_{s \leq t}$$

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Some (consistent and sequential) solutions...

- Linear+Gauss model : Kalman filters $\oplus$ Ensemble Kalman filters
- Nonlinear and/or non Gaussian models : Particle filters

In all cases . . .

$$p(\boldsymbol{X_t} \mid \boldsymbol{Y_t}) \;\propto\; p(\boldsymbol{Y_t} \mid \boldsymbol{X_t})\, p(\boldsymbol{X_t})$$

$\subset$ *Bayes, Kallianpur-Striebel, Feynman-Kac,. . .*

In all cases . . .

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"$\propto$" $\Longrightarrow$ **Nonlinear evolution equations (in the space of probab.)**

- Kalman filters $\supset$ Riccati equation
- In any case : Nonlinear filtering $\rightsquigarrow$ Nonlinear Markov processes

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Nonlinear Markov process

Generator/transitions depending on the law of internal states

Mean field particle samplers

- Sample multiple (interacting) copies
- Use the empirical distribution when needed

Important Observations & Questions

- Kalman/Riccati : Stable equations + Can stabilize unstable signals!
  Under appropriate observability and controllability conditions
- Nonlinear filtering equation : $\rightsquigarrow$ Stable (i.e. forget initial conditions)
  When the signal is stable/mixing $\rightsquigarrow$ *Stabilize unstable signals ?*

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Mean field (approximate) particle samplers

- (Nonlinear) Particle filters:
  Uniform estimates (w.r.t. time) when the signal is stable/mixing
- Ensemble Kalman filters:
  Stable equations + Can stabilize unstable signals!

Kalman-Bucy filter

$$\left\{ \begin{array}{rcl} dX_t &=& A\ X_t\ dt\ +\ R^{1/2}\ dW_t \\ dY_t &=& C\ X_t\ dt\ +\ \Sigma^{1/2}\ dV_t \end{array} \right. \Longrightarrow \eta_t = \mathcal{N}(\widehat{X}_t, P_t)$$

with the conditional mean/covariance matrix:

$$\widehat{X}_t := \mathbb{E}(X_t \mid \boldsymbol{Y_t}) \quad \text{and} \quad P_t := \mathbb{E}\left(\left(X_t - \widehat{X}_t\right)\left(X_t - \widehat{X}_t\right)'\right)$$

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Evolution equations

$$\begin{array}{rcl} d\widehat{X}_t & = & A\ \widehat{X}_t\ dt + P_t\ C'\Sigma^{-1}\ \left(dY_t - C\widehat{X}_t\ dt\right) \\ \partial_t P_t & = & \mathrm{Ricc}(P_t) := AP_t + P_t A' - P_t \mathbf{S} P_t + R \quad \text{with} \quad \mathbf{S} := \mathbf{C}'\boldsymbol{\Sigma}\mathbf{C} \end{array}$$

Nonlinear Kalman-Bucy diffusion

Nonlinear diffusions $\overline{X}_t$ depending on the (conditional) distributions

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Interaction with

$$\eta_t(e) \quad \text{and} \quad \mathcal{P}_{\eta_t} = \eta_t\left[(e - \eta_t(e))(e - \eta_t(e))'\right] \quad \text{with} \quad e(x) := x$$

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Consistency property

$$\eta_t := \mathcal{N}\left[\widehat{X}_t, P_t\right]$$

A couple of of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\, d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\,\left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\, d\overline{V}_t\right)\right]$$

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2) "deterministic EnKF" ($\rightsquigarrow$ discrete time - Sakov-Oke 08)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\, d\overline{W}_t + \mathcal{P}_{\eta_t} C'\Sigma^{-1} \left[dY_t - C\left(\frac{\overline{X}_t + \eta_t(e)}{2}\right) dt\right]$$

The Ensemble Kalman-Bucy filter

(Case 1) Mean field interpretation $\rightsquigarrow$ $N+1$ interacting diffusions

$$d\xi_t^i = A\,\xi_t^i dt + R^{1/2} d\overline{W}_t^i + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i dt + \Sigma^{1/2}\, d\overline{V}_t^i \right) \right]$$

with the rescaled particle covariance matrices

$$p_t := \left(1 + \frac{1}{N}\right) \mathcal{P}_{\eta_t^N} = \frac{1}{N} \sum_{1 \le i \le N+1} \left(\xi_t^i - m_t\right) \left(\xi_t^i - m_t\right)'$$

and the empirical measures

$$\eta_t^N := \frac{1}{N+1} \sum_{1 \le i \le N+1} \delta_{\xi_t^i} \quad \text{and the sample mean} \quad m_t := \frac{1}{N+1} \sum_{1 \le i \le N+1} \xi_t^i.$$

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Where are the Kalman & Riccati equations ?

Th: EnKF eq. [+ Tugaut (Arxiv-16/AAP-18)]

The EnKF (state estimate) equation

$$dm_t = A\ m_t dt + p_t\ C'\Sigma^{-1}\ (dY_t - Cm_t\ dt) + \frac{1}{\sqrt{N+1}}\ d\overline{M}_t$$

with an **r**-martingale $\overline{M}_t = \left(\overline{M}_t(k)\right)_{1\leq k\leq r}$ with angle brackets

$$\partial_t \langle \overline{M} \mid \otimes \mid \overline{M} \rangle_t = U + p_t V p_t.$$

With

$$1) \quad (U,V) = (R,S) \quad \text{and} \quad 2) \quad (U,V) = (R,0)$$

The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

Symmetric matrix-valued martingale $M_t = (M_t(k,l))_{1\leq k,l\leq r}$

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Orthogonality property

$$\forall 1 \leq k, l, l' \leq r \qquad \left\langle M(k,l), \overline{M}(l') \right\rangle_t = 0.$$

The 1d case $\rightsquigarrow$ Closed form Riccati semigroups

Deterministic Riccati P_t on $\mathbb{R}_+$: $\mathrm{Ricc}(\varpi_\pm) = 0$ for

$$S\varpi_- := A - \lambda/2 < 0 < S\varpi_+ := A + \lambda/2$$

with

$$\lambda = 2\sqrt{A^2 + RS}$$

$$\Downarrow$$

$\forall t \geq \upsilon > 0$

$$\left[|P_t - \varpi_+| \vee exp\left(2\int_0^t [A - P_s S] ds\right)\right] \leq c_\upsilon \ \exp\left(-\lambda t\right)$$

Stochastic Riccati flow $p_t \in \mathbb{R}_+$ with $\epsilon = 2/\sqrt{N}$:

$$dp_t \stackrel{law}{=} \mathrm{Ricc}(p_t)dt + \epsilon \sqrt{p_t(U + p_t V p_t)}\, dW_t$$

$$\text{with} \quad \epsilon^2 U < 2R \quad \Longrightarrow \quad \text{origin repellent}$$

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Reversible measures $\pi_\epsilon(dx)$ on $\mathbb{R}_+$:

- $U \wedge V > 0 \rightsquigarrow$ Heavy tails

$$\propto \quad \frac{x^{\frac{2}{\epsilon^2}\frac{R}{U}-1}}{[U + Vx^2]^{1+\frac{1}{\epsilon^2}\left(\frac{R}{U}+\frac{S}{V}\right)}}\ \exp\left[\frac{4}{\epsilon^2}\ \frac{A}{\sqrt{UV}}\ \tan^{-1}\left(x\sqrt{\frac{V}{U}}\right)\right]\ dx.$$

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- $U > V = 0 \rightsquigarrow$ Gaussian-type tails

$$\propto\ x^{\frac{2}{\epsilon^2}\frac{R}{U}-1}\ \exp\left[-\frac{S}{U\epsilon^2}\ \left(x - 2\ \frac{A}{S}\right)^2\right]\ dx.$$

Stability Markov transition semigroup $\mathcal{P}_t^\epsilon$ (of p_t)

Th [+ Bishop, Kamatani, Rémillard Arxiv-17] $\forall A, R \wedge S > 0$

- $\exists \zeta, \epsilon_0 > 0$ and some Wasserstein distance $\mathbb{D}$ s.t. for any $0 \leq \epsilon \leq \epsilon_0$

$$\mathbb{D}(\mu_1 \mathcal{P}_t^\epsilon, \mu_2 \mathcal{P}_t^\epsilon) \leq \exp\left(-\lambda \left(1 - \epsilon^2 \zeta\right) t\right) \; \mathbb{D}(\mu_1, \mu_2)$$

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$$\mathbb{E}\left[\exp\left[n \int_0^t (A - p_s S) ds\right]\right]^{1/n} \leq c_Q \; \exp\left(-\lambda \left(1 - \epsilon^2 \zeta_n\right) t\right)$$

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Some extensions

Case 2: Poincaré inequalities (and $\mathbb{L}_2(\pi_\epsilon)$-contractions), ...

Consequences

Uniform estimates for state estimates + particle Riccati diffusions,...

Multivariate KB : Observability + Controllability

$$d(\overline{X}_t - X_t)$$

$$= (\mathbf{A} - \mathbf{P_t S})\ (\overline{X}_t - X_t)dt\ +\ R^{1/2}\ d(\overline{W}_t - W_t) + P_t C' \Sigma^{-1/2}\ d\left[V_t - \overline{V}_t\right]$$

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Steady state: $\exists! P > 0$ **such that** $\mathrm{Ricc}(P) = 0$ **and spectral abscissa**

$$\varsigma(\mathbf{A} - \mathbf{PS}) := \max\left\{\mathrm{Re}(\lambda)\ :\ \lambda \in \mathrm{Spec}(A - PS)\right\} < 0$$

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$$\Downarrow$$

STABLE EVEN WHEN A is unstable.

$\rightsquigarrow$ `SIAM Control & Opt.-17` $\oplus$ `Arxiv-18` (+ Bishop)
Review on the stability of Kalman-Bucy filters and Riccati matrix semigroups $\oplus$ *Floquet representation of exponential semigroups*

Multivariate : EnKF

$$(m_t, X_t, \mathbf{p_t}) = (\text{sample mean}, \text{true signal}, \textbf{sample covariance})$$

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Observations:

- **Time varying** $\oplus$ **stochastic type** Ornstein-Uhlenbeck diffusion

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- The matrix $(A - pS)$ may be ill-conditioned in the sense that

  $$\exists p\ :\ \lambda_{max}((A - pS)_{sym}) > 0 > \lambda_{min}((A - pS)_{sym})$$

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- Always under-biased

$$\forall t > 0 \qquad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) < P_t$$

When A stable and $S > 0$

Theo [+ Tugaut (Arxiv-16/AAP-18)] $\forall n \geq 1\ \exists N_n \geq 1 : \forall N \geq N_n$

$$\sup_{t \geq 0} \left[\mathbb{E}\left(\|p_t - P_t\|^n\right)^{1/n} \vee \mathbb{E}(\|m_t - \widehat{X}_t\|^n)^{1/n} \right] < c_n/\sqrt{N}$$

Under : Observability + Controllability

Some additional and more refined theorems (N large enough)

Uniform moments + Uniform Riccati estimates + stability and invariant measures Riccati diffusions + non asymptotic CLT rates+ Bias-Taylor type expansions + Robustness and Perturbations analysis (inflation, masking, shrinkage, projections),. . .

- `SPA-17 (+ Bishop, Pathiraja):`
  Uniform robustness properties : inflation and localisation techniques.
- `Arxiv-17 (+ Bishop, Niclas):`
  Exact propagation of chaos expansions matrix Riccati diffusions, non asymptotic bias + CLT.
- `Arxiv-18 (+ Bishop):`
  Stability of stochastic Ornstein-Uhlenbeck diffusions
- in a few days/weeks (+ Bishop):
  Stability of Matrix Riccati equations.