Discrepancy of digital sequences: new results on a classical QMC topic

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Uniform distribution

Consider sequences

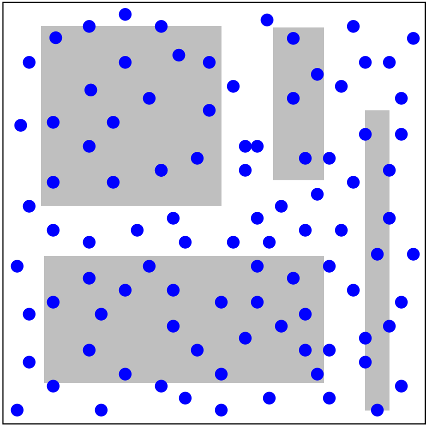
$$\mathcal{S} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots) \quad \text{in} \quad [0,1)^s.$$

For $N \in \mathbb{N}$ let $\mathcal{S}_N = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1})$.

Hermann Weyl 1916

$\mathcal{S}$ is **uniformly distributed** if for all axes parallel boxes $J \subseteq [0,1)^s$:

$$\lim_{N\to\infty} \frac{\#(\mathcal{S}_N \cap J)}{N} = \mathrm{Vol}(J).$$



Uniform distribution

Hermann Weyl 1916

Equivalent:

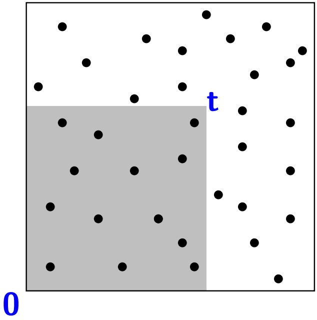
1. $\mathcal{S}$ is uniformly distributed;
2. for every Riemann-integrable $f : [0,1]^s \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

L_p-discrepancy

For $\mathcal{S}_N = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1})$ the **local discrepancy** is

$$\Delta_{\mathcal{S}_N}(\mathbf{t}) := \frac{\#(\mathcal{S}_N \cap [\mathbf{0}, \mathbf{t}))}{N} - \mathrm{Vol}([\mathbf{0}, \mathbf{t})); \quad \mathbf{t} \in [0,1]^s.$$



L_p-discrepancy of $\mathcal{S}$:

For $p \in [1, \infty]$ and $N \in \mathbb{N}$:

$$L_{p,N}(\mathcal{S}) = \|\Delta_{\mathcal{S}_N}\|_{L_p([0,1]^s)}$$

For $p = \infty$: $L_{\infty,N} = D_N^*$ (**star-discrepancy**)

$\mathcal{S}$ uniformly distributed $\Leftrightarrow$ $\lim_{N \to \infty} L_{p,N}(\mathcal{S}) = 0$

Discrepancy and QMC

Koksma-Hlawka inequality 1961

Let $f : [0,1]^s \to \mathbb{R}$ be with bounded variation $V(f)$ in the sense of Hardy and Krause. Then, for $\mathcal{S}_N = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1})$,

$$\left| \int_{[0,1]^s} f(\mathbf{x}) \,\mathrm{d}\mathbf{x} - \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right| \le V(f) D_N^*(\mathcal{S}_N).$$

Known results (finite sequences)

Roth (1954); Schmidt (1977); Bilyk, Lacey, Vagharshakyan (2008)

For every $p \in (1, \infty]$ for every finite sequence $\mathcal{S}_N$ in $[0,1)^s$

$$L_{p,N}(\mathcal{S}_N) \geq c_{p,s} \frac{(\log N)^{\frac{s-1}{2}}}{N} \quad \text{and} \quad D_N^*(\mathcal{S}_N) \geq c_{\infty,s} \frac{(\log N)^{\frac{s-1}{2}+\eta_s}}{N}$$

for some $\eta_s \in (0, \frac{1}{2})$.

Halász (1981); Schmidt (1972)

For $s = 2$ for every $\mathcal{S}_N$ in $[0,1)^2$

$$L_{1,N}(\mathcal{S}_N) \geq c_{1,2} \frac{\sqrt{\log N}}{N} \quad \text{and} \quad D_N^*(\mathcal{S}_N) \geq c_{\infty,2} \frac{\log N}{N}.$$

Known results (finite sequences)

- There exist $\mathcal{S}_N$ in $[0,1)^s$:

$$D_N^*(\mathcal{S}_N) \lesssim_s \frac{(\log N)^{s-1}}{N}.$$

Example: Hammersley-net (1960), digital nets (Niederreiter 1987)

- There exist $\mathcal{S}_N$ in $[0,1)^s$:

$$L_{p,N}(\mathcal{S}_N) \lesssim_{s,p} \frac{(\log N)^{\frac{s-1}{2}}}{N}.$$

Example: Chen & Skriganov (2002), Skriganov (2006), Dick & Pill. (2014), Markhasin (2015), ...

Finite vs. infinite sequences

Conceptual difference between $L_{p,N}(\mathcal{S}_N)$ and $L_{p,N}(\mathcal{S})$ (Matoušek 1999):

discrepancy of finite $\mathcal{S}_N$	discrepancy of infinite $\mathcal{S}$
static setting	dynamic setting
N fixed	$N = 1, 2, 3, 4, \ldots$
behavior of the whole set	behavior of **all** initial segments
$(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1})$	$(\mathbf{x}_0)$, $(\mathbf{x}_0, \mathbf{x}_1)$,…, $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1})$

Known results (infinite sequ.); Method of Proĭnov 1985

For every $p \in (1, \infty]$ for every infinite sequence $\mathcal{S}$ in $[0,1)^s$

$$L_{p,N}(\mathcal{S}) \geq c_{p,s} \frac{(\log N)^{\frac{s}{2}}}{N} \quad \text{infinitely often.}$$

For $p = \infty$ there exist $\eta_s \in (0, \frac{1}{2})$ such that for every $\mathcal{S}$

$$D_N^*(\mathcal{S}) \geq c_{\infty,s} \frac{(\log N)^{\frac{s}{2}+\eta_s}}{N} \quad \text{infinitely often.}$$

For $s = 1$ for every $\mathcal{S}$ in $[0,1)$

$$L_{1,N}(\mathcal{S}) \geq c_{1,1} \frac{\sqrt{\log N}}{N} \quad \text{infinitely often}$$

and

$$D_N^*(\mathcal{S}) \geq c_{\infty,1} \frac{\log N}{N} \quad \text{infinitely often.}$$

Known results and open questions

Grand conjecture

For every $\mathcal{S}$ in $[0,1)^s$: $D_N^*(\mathcal{S}) \geq c_s \frac{(\log N)^s}{N}$ infinitely often.

There exist $\mathcal{S}$ in $[0,1)^s$ such that

$$D_N^*(\mathcal{S}) \lesssim_s \frac{(\log N)^s}{N} \quad \text{for all } N \geq 2.$$

Example: van der Corput sequence (1935), Halton sequence (1960), Sobol' sequence (1967), Faure sequence (1982), ...

Question

For $p < \infty$: Are there sequences $\mathcal{S}$ in $[0,1)^s$ such that

$$L_{p,N}(\mathcal{S}) \lesssim_{p,s} \frac{(\log N)^{\frac{s}{2}}}{N} \quad \text{for all } N \geq 2.$$

U.D. sequences with low discrepancy

- **Kronecker sequences:** $\mathcal{S}(\boldsymbol{\alpha}) = (\{n\boldsymbol{\alpha}\})_{n\geq 0}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$:

  $\mathcal{S}(\boldsymbol{\alpha})$ u.d. $\quad\Leftrightarrow\quad$ $1, \alpha_1, \ldots, \alpha_s$ linearly independent over $\mathbb{Q}$.

- **Digital sequences:**
  - **van der Corput** sequence (1935)
  - pioneering work by **Sobol'** (1967) and **Faure** (1982)

Faure, Henri: Discrépance de suites associées à un système de numération (en dimension s). Acta Arith. 41 (1982), no. 4, 337–351.

  - general concept **Niederreiter** (1987)

Digital sequences

Definition (Niederreiter 1987)

- Let $b \in \mathbb{P}$ and let $\mathbb{F}_b$ be the finite field of order b;
- choose $C_1, \ldots, C_s \in \mathbb{F}_b^{\mathbb{N}\times\mathbb{N}}$;
- for $n \in \mathbb{N}_0$ of the form $n = n_0 + n_1 b + n_2 b^2 + \cdots$ compute (over $\mathbb{F}_b$)

$$C_j \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \end{pmatrix} =: \begin{pmatrix} x_{n,j,1} \\ x_{n,j,2} \\ x_{n,j,3} \\ \vdots \end{pmatrix};$$

- put

$$x_{n,j} = \frac{x_{n,j,1}}{b} + \frac{x_{n,j,2}}{b^2} + \frac{x_{n,j,3}}{b^3} + \cdots \quad \text{and} \quad \mathbf{x}_n = (x_{n,1}, \ldots, x_{n,s}).$$

- $\mathcal{S}(C_1, \ldots, C_s) = (\mathbf{x}_n)_{n \geq 0}$ is called a **digital sequence over** $\mathbb{F}_b$.

Digital Kronecker sequences

Field of **formal Laurent series** over $\mathbb{F}_b$ in the variable t:

$$\mathbb{F}_b((t^{-1})) = \left\{ \sum_{i=w}^{\infty} g_i t^{-i} \ : \ w \in \mathbb{Z}, \forall i : g_i \in \mathbb{F}_b \right\}.$$

For $g = \sum_{i=w}^{\infty} g_i t^{-i}$ define

$$\{g\} := \sum_{i=\max\{w,1\}}^{\infty} g_i t^{-i}.$$

Let $n \in \mathbb{N}_0$ with b-adic expansion

$$n = n_0 + n_1 b + \cdots + n_r b^r, \quad \text{where} \quad n_i \in \{0, \ldots, p-1\},$$

then

$$n \cong n_0 + n_1 t + \cdots + n_r t^r \in \mathbb{F}_b[t].$$

Digital Kronecker sequences

Definition

Let $\mathbf{f} = (f_1, \ldots, f_s) \in \mathbb{F}_b((t^{-1}))^s$. Then the sequence $\mathcal{S}(\mathbf{f}) = (\mathbf{y}_n)_{n \geq 0}$ given by

$$\mathbf{y}_n := \{n\mathbf{f}\}_{|t=b} = (\{nf_1\}_{|t=b}, \ldots, \{nf_s\}_{|t=b})$$

is called a **digital Kronecker sequence over** $\mathbb{F}_b$.

A digital Kronecker sequence is a digital sequence where

$$C_j = \begin{pmatrix} f_{j,1} & f_{j,2} & f_{j,3} & \iddots \\ f_{j,2} & f_{j,3} & f_{j,4} & \iddots \\ f_{j,3} & f_{j,4} & f_{j,5} & \iddots \\ \iddots & \iddots & \iddots & \iddots \end{pmatrix} \quad \text{for } f_j = \frac{f_{j,1}}{t} + \frac{f_{j,2}}{t^2} + \frac{f_{j,3}}{t^3} + \frac{f_{j,4}}{t^4} + \cdots.$$

A metrical discrepancy bound

Theorem (Larcher 1998, Larcher & Pill. 2014)

Let $\varepsilon > 0$. For almost all s-tuples $(C_1, \ldots, C_s)$ with $C_j \in \mathbb{F}_b^{\mathbb{N}\times\mathbb{N}}$ the corresponding digital sequences $\mathcal{S} = \mathcal{S}(C_1, \ldots, C_s)$ satisfy

$$D_N^*(\mathcal{S}) \lesssim_{b,s,\varepsilon} \frac{(\log N)^s (\log \log N)^{2+\varepsilon}}{N} \quad \forall N \geq 2$$

and

$$D_N^*(\mathcal{S}) \geq c_{b,s} \frac{(\log N)^s \log \log N}{N} \quad \text{infinitely often.}$$

A metrical discrepancy bound (digital Kronecker sequ.)

Theorem (Larcher 1995, Larcher & Pill. 2014)

Let $\varepsilon > 0$. For almost all $\mathbf{f} \in \mathbb{F}_b((t^{-1}))^s$ the corresponding digital Kronecker sequences $\mathcal{S} = \mathcal{S}(\mathbf{f})$ satisfy

$$D_N^*(\mathcal{S}) \lesssim_{b,s,\varepsilon} \frac{(\log N)^s (\log\log N)^{2+\varepsilon}}{N} \quad \forall N \geq 2$$

and

$$D_N^*(\mathcal{S}) \geq c_{b,s} \frac{(\log N)^s \log\log N}{N} \quad \text{infinitely often.}$$

Classical Kronecker sequences: **Joszef Beck** (1994).

Digital (t, s)-sequences

For $m \in \mathbb{N}$ denote by $C(m)$ the left upper $m \times m$ submatrix of C.

Definition (Niederreiter)

Given $C_1, \ldots, C_s$. If $\exists\, t \in \mathbb{N}_0$: for every $m \geq t$ and for all $d_1, \ldots, d_s \geq 0$ with $d_1 + \cdots + d_s = m - t$ the

$$\left.\begin{array}{l} \text{first } d_1 \text{ rows of } C_1(m), \\ \text{first } d_2 \text{ rows of } C_2(m), \\ \ldots \\ \text{first } d_s \text{ rows of } C_s(m), \end{array}\right\} \text{ are linearly independent over } \mathbb{F}_b,$$

then $\mathcal{S}(C_1, \ldots, C_s)$ is called a **digital (t, s)-sequence over $\mathbb{F}_b$**.

Examples: generalized Niederreiter sequences (Sobol', Faure, orig. Niederreiter), Niederreiter-Xing sequences, ...

Star discrepancy of digital (t, s)-sequences

Every sub-block $(\mathbf{x}_{kb^m}, \mathbf{x}_{kb^m+1}, \ldots, \mathbf{x}_{(k+1)b^m-1})$ is a (t, m, s)-net, i.e., every

$$J = \prod_{j=1}^{s} \left[\frac{a_j}{b^{d_j}}, \frac{a_j + 1}{b^{d_j}} \right) \qquad \text{with } \mathrm{Vol}(J) = b^{t-m}$$

contains the right share $(= b^t)$ of elements.

Theorem (Niederreiter 1987)

For every digital (t, s)-sequence $\mathcal{S}$ over $\mathbb{F}_b$ we have

$$D_N^*(\mathcal{S}) \leq c_{s,b}\, b^t \frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N} \right).$$

Further results:

- **Faure & Kritzer** (2013) . . . . . . smallest $c_{s,b}$
- **Faure & Lemieux** (2012, 2014, 2105)
- **Tezuka** (2013)

Star discrepancy of digital (t,s)-sequences

Levin: $(\mathbf{x}_n)_{n\ge 0}$ in $[0,1)^s$ is called d-admissible if

$$\inf_{n>k\ge 0} \|n \ominus k\|_b \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b \ge b^{-d},$$

where $\log_b \|x\|_b = \lfloor \log_b x \rfloor$ and $\ominus$ the b-adic difference.

Theorem (Levin 2017)

Let $\mathcal{S}$ be a d-admissible (t,s)-sequence. Then

$$D_N^*(\mathcal{S}) \ge c_{s,t,d} \frac{(\log N)^s}{N} \quad \text{infinitely often.}$$

Examples:

- generalized Niederreiter sequences (Sobol', Faure, orig. Niederreiter)
- Niederreiter-Xing sequences

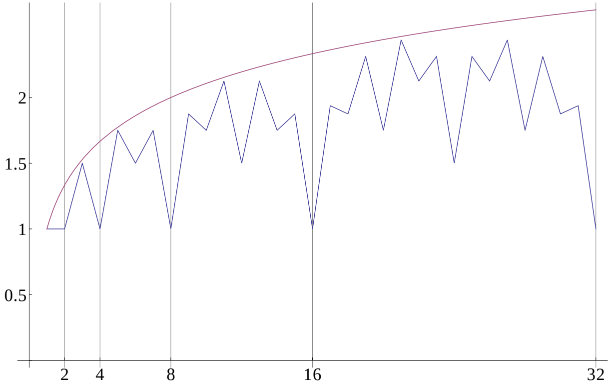
Levin: positive support for the grand conjecture in discrepancy theory

L_p discrepancy of digital $(0,1)$-sequences

Van der Corput sequence $\mathcal{S}(I)$ with $\mathbb{N} \times \mathbb{N}$ identity matrix I:

$$D_N^*(\mathcal{S}(C)) \le D_N^*(\mathcal{S}(I)) \le \begin{cases} \left(\frac{\log N}{3 \log 2} + 1\right) \frac{1}{N}; \\ \frac{S_2(N)}{N} \ \ldots \text{dyadic sum-of-digits fct.} \end{cases}$$

and also $L_{2,N}(\mathcal{S}(C)) \le L_{2,N}(\mathcal{S}(I))$.



L_p discrepancy of digital $(0,1)$-sequences

- (Pill. 2004) Van der Corput sequence For all $p \in [1,\infty)$

$$\limsup_{N\to\infty} \frac{N L_{p,N}(\mathcal{S}(I))}{\log N} = \frac{1}{6\log 2}.$$

- (Drmota, Larcher & Pill. 2005) For $C = \begin{pmatrix} 1 & 1 & 1 & \ldots \\ 0 & 1 & 1 & \ldots \\ 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$

$$\limsup_{N\to\infty} \frac{N L_{2,N}(\mathcal{S}(C))}{\log N} \geq c > 0.$$

In general digital (t,s)-sequences do not achieve $L_p(\mathcal{S}) \lesssim \frac{1}{N}(\log N)^{\frac{s}{2}}$.

▶ we need additional more demanding properties:

Higher order digital sequences (Josef Dick 2007)

Explicit constructions of order 2 digital sequences

- Let $C_1, \ldots, C_{2s}$ generate a digital $(t', 2s)$-sequence over $\mathbb{F}_2$.
  Example: generalized Niederreiter sequence
- **Interlacing:**

$$C_1 = \begin{pmatrix} \vec{c}_{1,1} \\ \vec{c}_{1,2} \\ \vdots \end{pmatrix}, C_2 = \begin{pmatrix} \vec{c}_{2,1} \\ \vec{c}_{2,2} \\ \vdots \end{pmatrix} \rightarrow E_1 = \begin{pmatrix} \vec{c}_{1,1} \\ \vec{c}_{2,1} \\ \vec{c}_{1,2} \\ \vec{c}_{2,2} \\ \vdots \end{pmatrix}$$

Theorem (Dick 2007)

The digital sequence generated by $E_1, \ldots, E_s$ is an **order 2 digital (t,s)-sequence over $\mathbb{F}_2$** with $t = 2t' + s$.

L_p-discrepancy of order 2 digital sequences

Theorem (Dick, Hinrichs, Markhasin, Pill. 2017)

For every $p \in [1, \infty)$ and every order 2 digital (t, s)-sequence $\mathcal{S}$ over $\mathbb{F}_2$ + additional property we have

$$L_{p,N}(\mathcal{S}) \lesssim_{p,s} 2^t \frac{(\log N)^{\frac{s}{2}}}{N} \quad \text{for all} \quad N \geq 2.$$

Theorem (Dick & Pill. 2014)

Explict construction of order **5** digital sequence over $\mathbb{F}_2$:

$$L_{2,N}(\mathcal{S}) \lesssim_s \frac{(\log N)^{\frac{s-1}{2}}}{N} \sqrt{S_2(N)} \quad \text{for all} \quad N \geq 2.$$

$\uparrow$ dyadic sum-of-digits function

L_p-discrepancy of order 2 digital sequences

Littlewood-Paley type estimate

Let $p \in (1,\infty)$, $\bar{p} = \max(p,2)$ and $h_{\mathbf{j},\mathbf{m}}$ are the Haar functions on $[0,1)^s$.

$$\|f\|_{L_p([0,1)^s)}^2 \lesssim_{p,s} \sum_{\mathbf{j}\in\mathbb{N}_{-1}^s} 2^{2|\mathbf{j}|(1-1/\bar{p})} \left(\sum_{\mathbf{m}\in\mathbb{D}_\mathbf{j}} |\langle f, h_{\mathbf{j},\mathbf{m}}\rangle|^{\bar{p}} \right)^{2/\bar{p}} .$$

Likewise: quasi-norm of the discrepancy function in Besov spaces and Triebel-Lizorkin spaces with dominating mixed smoothness:

- **Triebel** (2010)
- **Hinrichs** (2010)
- **Markhasin** (2013, 2015)
- **Kritzinger** (2016, 2018)
- **DHMP** (2017) ▶ Matching lower and upper bounds for order 2 DS

Intermediate norms

Gap of knowledge between L_p discrepancy for finite vs. infinite p:

- $p < \infty$:

$$\min L_{p,N} \asymp \frac{(\log N)^{\frac{s}{2}}}{N}$$

- $p = \infty$:

$$\frac{(\log N)^{\frac{s}{2}+\eta_s}}{N} \lesssim \min D_N^* \lesssim \frac{(\log N)^s}{N}$$

What happens in intermediate spaces

between L_p for $p < \infty$ and L_∞ "close" to L_∞?

Intermediate norms

Examples:

- exponential Orlicz norm: for any $\beta > 0$

$$\|f\|_{\exp(L^\beta)} \asymp_s \sup_{p>1} p^{-\frac{1}{\beta}} \|f\|_{L_p([0,1]^s)}.$$

- BMO (seminorm)

$$\|f\|_{\mathrm{BMO}^s}^2 = \sup_{U \subseteq [0,1)^s} \frac{1}{\lambda_s(U)} \sum_{\mathbf{j} \in \mathbb{N}_0^s} 2^{|\mathbf{j}|} \sum_{\substack{\mathbf{m} \in \mathbb{D}_\mathbf{j} \\ \mathrm{supp}(h_{\mathbf{j},\mathbf{m}}) \subseteq U}} |\langle f, h_{\mathbf{j},\mathbf{m}} \rangle|^2.$$

BMO-discrepancy of $\mathcal{S}$

$$L_{\mathrm{BMO},N}(\mathcal{S}) := \|\Delta_{\mathcal{S}_N}\|_{\mathrm{BMO}^s}$$

Results for finite point sets

- **Bilyk, Lacey, Parissis, Vagharshakyan** (2009)
- **Bilyk & Markhasin** (2016)

BMO-discrepancy

Theorem (DHMP 2017)

For every $\mathcal{S}$ in $[0,1)^s$ we have

$$L_{\mathrm{BMO},N}(\mathcal{S}) \geq c_s \frac{(\log N)^{\frac{s}{2}}}{N} \quad \text{infinitely often.}$$

Theorem (DHMP 2017)

Let $\mathcal{S}$ be an order 2 digital (t,s)-sequence over $\mathbb{F}_2$ + additional property. Then we have

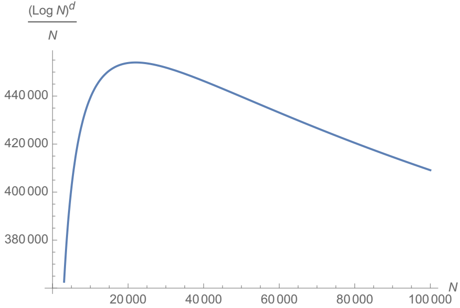
$$L_{\mathrm{BMO},N}(\mathcal{S}) \lesssim_s \frac{(\log N)^{\frac{s}{2}}}{N} \quad \text{for all} \quad N \geq 2.$$

Discussion of the asymptotic bounds

Discrepancy/QMC can achieve error bounds of order of magnitude

$$D_N^* \lesssim \frac{(\log N)^s}{N}.$$

Problem: $N \mapsto (\log N)^s/N$ is increasing for $N \le \mathrm{e}^s$



E.g. $s = 200$, then $\mathrm{e}^s \approx 7.2 \times 10^{86}$.

Viewpoint of "Information Based Complexity"

Study the dependence of the worst-case error on the dimension s

Theorem (Heinrich, Novak, Wasilkowski, Woźniakowski 2001)

For all $N, s \in \mathbb{N}$ there exist $\mathcal{S}_N$ in $[0,1)^s$ such that

$$D_N^*(\mathcal{S}_N) \lesssim_{\mathrm{abs}} \sqrt{\frac{s}{N}}.$$

- $N^*(\varepsilon, s) = \min\{N \ : \ \exists \mathcal{S}_N \subseteq [0,1)^s \text{ s.t. } D_N^*(\mathcal{S}_N) \le \varepsilon\} \lesssim s\varepsilon^{-2}$.
- IBC: polynomial tractability
- **Hinrichs:** $N^*(\varepsilon, s) \ge cs\varepsilon^{-1}$ for all $\varepsilon \in (0, \varepsilon_0)$ and $s \in \mathbb{N}$
- exact dependence of $N^*(\varepsilon, s)$ on ε^{-1} still open
- **Aistleitner:** $D_N^*(\mathcal{S}_N) \le 10\sqrt{s/N}$.

A metrical result

Definition

For $\mathbf{f} \in \mathbb{F}_b((t^{-1}))^s$ consider $\mathcal{S}(\mathbf{f}) = (\mathbf{y}_n)_{n \geq 0}$ where

$$\mathbf{y}_n = \{t^n \mathbf{f}\}_{|t=b} = (\{t^n f_1\}_{|t=b}, \ldots, \{t^n f_s\}_{|t=b}).$$

Theorem (Neumüller & Pill. 2016)

Let $s \geq 2$. For every $\delta \in (0,1)$ we have

$$D_N^*(\mathcal{S}(\mathbf{f})) \lesssim_{b,\delta} \sqrt{\frac{s \log s}{N}} \log N \quad \text{for all } N \geq 2$$

with probability at least $1-\delta$, where the implied constant $C_{b,\delta} \asymp_b \log \delta^{-1}$.

Classical Kronecker sequences: **Löbbe** (2014)

Weighted star discrepancy

Let $\gamma_1, \gamma_2, \gamma_3, \ldots \in \mathbb{R}^+$ and put

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j \quad \text{for} \quad \emptyset \neq \mathfrak{u} \subseteq [s].$$

Weighted star discrepancy (Sloan & Woźniakowski 1998)

$$D^*_{N,\gamma}(\mathcal{S}) = \max_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} D^*_N(\mathcal{S}(\mathfrak{u})).$$

If $\gamma_j = 1$ for all $j \geq 1$, then

$$D^*_{N,\gamma}(\mathcal{S}) = D^*_N(\mathcal{S}).$$

Weighted star discrepancy and multivariate integration

Consider

$$\mathcal{F}_{s,1,\gamma} = \{f : [0,1]^s \to \mathbb{R} \;:\; \|f\|_{s,1,\gamma} < \infty\},$$

where

$$\|f\|_{s,1,\gamma} = |f(\mathbf{1})| + \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \frac{1}{\gamma_{\mathfrak{u}}} \left\| \frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}_{\mathfrak{u}}, \mathbf{1}) \right\|_{L_1}.$$

- small $\gamma_{\mathfrak{u}}$ forces $\|\frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}_{\mathfrak{u}}, \mathbf{1})\|_{L_1}$ to be small in order to guarantee $\|f\|_{\mathcal{F}_{s,1,\gamma}} \leq 1$

Theorem (Sloan & Woźniakowski 1998)

$$\mathrm{wce}(\mathcal{F}_{s,1,\gamma}, \mathcal{S}_N) = D^*_{N,\gamma}(\mathcal{S}_N)$$

Weighted star discrepancy and tractability

Let

$$N_{\min}(\varepsilon, s) := \min\{N \ : \ \exists \mathcal{S}_N \subseteq [0,1)^s \text{ s.t. } D^*_{N,\gamma}(\mathcal{S}_N) \leq \varepsilon\}.$$

The weighted star discrepancy is said to be

- **strongly polynomially tractable**, if there exist non-negative real numbers C and β such that

$$N_{\min}(\varepsilon, s) \leq C\varepsilon^{-\beta} \quad \text{for all } s \in \mathbb{N} \text{ and for all } \varepsilon \in (0,1). \tag{1}$$

  The infimum β^* over all $\beta > 0$ such that (1) holds is called the ε-exponent of strong polynomial tractability.

Weighted star discrepancy and tractability

Theorem (Hinrichs, Pill., Tezuka 2018)

The weighted star discrepancy of **Niederreiter sequences** achieves SPT with

- $\beta^* = 1$, which is optimal, if

$$\sum_{j \geq 1} j\gamma_j < \infty \qquad \text{e.g. } \gamma_j = \frac{1}{j^{2+\delta}};$$

- $\beta^* \leq 2$, if

$$\sup_{s \geq 1} \max_{\emptyset \neq \mathfrak{u} \subseteq [s]} \prod_{j \in \mathfrak{u}} (j\gamma_j) < \infty \qquad \text{e.g. } \gamma_j = \frac{1}{j}.$$

Weighted star discrepancy and tractability

Aistleitner (2014): If

$$\sum_{j=1}^{\infty} \exp(-c\gamma_j^{-2}) < \infty \qquad \text{e.g. } \gamma_j = \frac{1}{\sqrt{\log j}}$$

for some $c > 0$ then for all $s, N \in \mathbb{N}$ there exists a $\mathcal{S}_N$ in $[0,1)^s$:

$$D_{N,\gamma}^*(\mathcal{S}_N) \lesssim_\gamma \frac{1}{\sqrt{N}} \qquad \text{i.e., SPT with } \beta^* \le 2.$$

Open problem: construct $\mathcal{S}_N$

Summary

- Digital sequences: excellent discrepancy properties in an asymptotic sense:
  - $p \in [1, \infty)$:
    $$L_p(\mathcal{S}) \lesssim \frac{(\log N)^{s/2}}{N} \qquad \text{(best possible)}$$
  - $p = \infty$:
    $$D_N^*(\mathcal{S}) \lesssim \frac{(\log N)^s}{N} \qquad \text{(presumably best possible)}$$
  - best possible discrepancy bounds for many other norms of $\Delta_{\mathcal{S}}$
- Good properties also with respect to tractability of discrepancy, but here . . .

many **open questions**