Weak and strong approximation of fractional order elliptic equations with spatial white noise

David Bolin, **Mihály Kovács** and Kristin Kirchner

Mathematical Sciences, Chalmers University of Technology and University of Gothenburg

MCQMC2018, Rennes, 1-6 July, 2018

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0,1). \tag{1}$$

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0, 1). \tag{1}$$

- H is a separable Hilbert space

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0,1). \tag{1}$$

- H is a separable Hilbert space
- $L\colon \mathscr{D}(L) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator with a compact inverse

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0,1). \tag{1}$$

- H is a separable Hilbert space
- $L \colon \mathscr{D}(L) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator with a compact inverse
- $\dot{\mathcal{W}}$ denotes Gaussian white noise "in H" defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0,1). \tag{1}$$

- H is a separable Hilbert space
- $L\colon \mathscr{D}(L) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator with a compact inverse
- $\dot{\mathcal{W}}$ denotes Gaussian white noise "in H" defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$
- $g \in H$

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0, 1). \tag{1}$$

- H is a separable Hilbert space
- $L \colon \mathscr{D}(L) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator with a compact inverse
- $\dot{\mathcal{W}}$ denotes Gaussian white noise "in H" defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$
- $g \in H$
- u is an H-valued random variable

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0,1). \tag{1}$$

- H is a separable Hilbert space
- $L \colon \mathscr{D}(L) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator with a compact inverse
- $\dot{\mathcal{W}}$ denotes Gaussian white noise "in H" defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$
- $g \in H$
- u is an H-valued random variable
- Typically $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded spatial domain in $\mathbb{R}^d$ and L is an elliptic integer order differential operator

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0,1). \tag{1}$$

- H is a separable Hilbert space
- $L\colon \mathscr{D}(L) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator with a compact inverse
- $\dot{\mathcal{W}}$ denotes Gaussian white noise "in H" defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$
- $g \in H$
- u is an H-valued random variable
- Typically $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded spatial domain in $\mathbb{R}^d$ and L is an elliptic integer order differential operator
- Goal: discretize (1) in a computationally feasible way together with error estimates

Problem formulation

We consider the fractional order random PDE written in the abstract form

$$L^{\beta} u = g + \dot{\mathcal{W}}, \quad \beta \in (0,1). \tag{1}$$

- H is a separable Hilbert space
- $L \colon \mathscr{D}(L) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator with a compact inverse
- $\dot{\mathcal{W}}$ denotes Gaussian white noise "in H" defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$
- $g \in H$
- u is an H-valued random variable
- Typically $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded spatial domain in $\mathbb{R}^d$ and L is an elliptic integer order differential operator
- Goal: discretize (1) in a computationally feasible way together with error estimates
- Motivation: approximation of Gaussian Matérn fields via

  $$(\kappa^2 - \Delta)^{\beta} u(\mathbf{x}) = \dot{\mathcal{W}}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{D}$$

  supplemented by Neumann boundary conditions

Preliminaries

- Denote the eigenvalue-eigenvector pairs $\{(\lambda_j, e_j)\}_{j\in\mathbb{N}}$ with

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots, \qquad \lim_{j\to\infty} \lambda_j = \infty.$$

Preliminaries

- Denote the eigenvalue-eigenvector pairs $\{(\lambda_j, e_j)\}_{j\in\mathbb{N}}$ with
$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots, \qquad \lim_{j\to\infty} \lambda_j = \infty.$$
- Assume that the growth of the eigenvalues is given by $\lambda_j \sim j^\alpha$ for an exponent $\alpha > 0$, i.e., there exist constants $c_\lambda, C_\lambda > 0$ such that
$$c_\lambda\, j^\alpha \leq \lambda_j \leq C_\lambda\, j^\alpha \qquad \forall j \in \mathbb{N}.$$

Preliminaries

- Denote the eigenvalue-eigenvector pairs $\{(\lambda_j, e_j)\}_{j\in\mathbb{N}}$ with
$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots, \qquad \lim_{j\to\infty} \lambda_j = \infty.$$
- Assume that the growth of the eigenvalues is given by $\lambda_j \sim j^\alpha$ for an exponent $\alpha > 0$, i.e., there exist constants $c_\lambda, C_\lambda > 0$ such that
$$c_\lambda j^\alpha \le \lambda_j \le C_\lambda j^\alpha \qquad \forall j \in \mathbb{N}.$$
- For $\beta > 0$ and $\phi \in \mathscr{D}(L^\beta) := \{\psi \in H : \sum_{j\in\mathbb{N}} \lambda_j^{2\beta}(\psi, e_j)_H^2 < \infty\}$ the action of the fractional power operator $L^\beta : \mathscr{D}(L^\beta) \to H$ is defined by
$$L^\beta \phi := \sum_{j\in\mathbb{N}} \lambda_j^\beta (\phi, e_j)_H \, e_j.$$
The subspace $\dot{H}^{2\beta} := \mathscr{D}(L^\beta) \subset H$ is itself a Hilbert space with respect to the inner product and corresponding norm given by
$$(\phi, \psi)_{2\beta} := (L^\beta \phi, L^\beta \psi)_H, \qquad \|\phi\|_{2\beta}^2 := \|L^\beta \phi\|_H^2 = \sum_{j\in\mathbb{N}} \lambda_j^{2\beta} (\phi, e_j)_H^2.$$

Preliminaries

- Denote the eigenvalue-eigenvector pairs $\{(\lambda_j, e_j)\}_{j\in\mathbb{N}}$ with
$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots, \qquad \lim_{j\to\infty} \lambda_j = \infty.$$
- Assume that the growth of the eigenvalues is given by $\lambda_j \sim j^\alpha$ for an exponent $\alpha > 0$, i.e., there exist constants $c_\lambda, C_\lambda > 0$ such that
$$c_\lambda j^\alpha \le \lambda_j \le C_\lambda j^\alpha \qquad \forall j \in \mathbb{N}.$$
- For $\beta > 0$ and $\phi \in \mathscr{D}(L^\beta) := \{\psi \in H : \sum_{j\in\mathbb{N}} \lambda_j^{2\beta}(\psi, e_j)_H^2 < \infty\}$ the action of the fractional power operator $L^\beta : \mathscr{D}(L^\beta) \to H$ is defined by
$$L^\beta \phi := \sum_{j\in\mathbb{N}} \lambda_j^\beta (\phi, e_j)_H \, e_j.$$
The subspace $\dot{H}^{2\beta} := \mathscr{D}(L^\beta) \subset H$ is itself a Hilbert space with respect to the inner product and corresponding norm given by
$$(\phi, \psi)_{2\beta} := (L^\beta \phi, L^\beta \psi)_H, \qquad \|\phi\|_{2\beta}^2 := \|L^\beta \phi\|_H^2 = \sum_{j\in\mathbb{N}} \lambda_j^{2\beta} (\phi, e_j)_H^2.$$
- Its dual space is denoted by $\dot{H}^{-2\beta}$. For $s \ge 0$, the norm on the dual space $\dot{H}^{-s}$ enjoys the useful representation
$$\|g\|_{-s} = \sup_{\phi \in \dot{H}^s \setminus \{0\}} \frac{\langle g, \phi \rangle}{\|\phi\|_s} = \left(\sum_{j\in\mathbb{N}} \lambda_j^{-s} \langle g, e_j \rangle^2 \right)^{1/2},$$
where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\dot{H}^{-s}$ and $\dot{H}^s$.

Gaussian white noise and regularity

- The white noise $\dot{\mathcal{W}}$ can formally be represented by the Karhunen–Loève expansion with respect to the orthonormal eigenbasis $\{e_j\}_{j\in\mathbb{N}} \subset H$ of L:

$$\dot{\mathcal{W}} = \sum_{j\in\mathbb{N}} \xi_j \, e_j,$$

where $\{\xi_j\}_{j\in\mathbb{N}}$ is a sequence of independent real-valued standard normally distributed random variables.

Gaussian white noise and regularity

- The white noise $\dot{\mathcal{W}}$ can formally be represented by the Karhunen–Loève expansion with respect to the orthonormal eigenbasis $\{e_j\}_{j\in\mathbb{N}} \subset H$ of L:

$$\dot{\mathcal{W}} = \sum_{j\in\mathbb{N}} \xi_j \, e_j,$$

  where $\{\xi_j\}_{j\in\mathbb{N}}$ is a sequence of independent real-valued standard normally distributed random variables.

Lemma

It holds that $\dot{\mathcal{W}} \in L_2\big(\Omega; \dot{H}^{-\frac{1}{\alpha}-\epsilon}\big)$ for all $\epsilon > 0$ with

$$\mathbb{E}\big[\|\dot{\mathcal{W}}\|^2_{-\frac{1}{\alpha}-\epsilon}\big] \leq c_\lambda^{-\frac{1}{\alpha}-\epsilon} \left(1 + \tfrac{1}{\epsilon\alpha}\right).$$

Furthermore, the solution u of (1) satisfies $u \in L_2\big(\Omega; \dot{H}^{2\beta-\frac{1}{\alpha}-\epsilon}\big)$. In particular, $u \in L_2(\Omega; H)$ holds if $2\alpha\beta > 1$.

Approximation: Galerkin finite elements

- $(V_h)_{h\in(0,1)}$ of subspaces of $\dot{H}^1$ with finite dimensions $N_h := \dim(V_h) < \infty$

Approximation: Galerkin finite elements

- $(V_h)_{h \in (0,1)}$ of subspaces of $\dot{H}^1$ with finite dimensions $N_h := \dim(V_h) < \infty$
- $\Pi_h \colon H \to V_h$ denotes the H-orthogonal projection onto V_h

Approximation: Galerkin finite elements

- $(V_h)_{h\in(0,1)}$ of subspaces of $\dot{H}^1$ with finite dimensions $N_h := \dim(V_h) < \infty$
- $\Pi_h \colon H \to V_h$ denotes the H-orthogonal projection onto V_h
- We denote version of L by $L_h \colon V_h \to V_h$, i.e., $(L_h\psi_h, \phi_h)_H = (L^{1/2}\psi_h, L^{1/2}\phi_h)_H$ for all $\psi_h, \phi_h \in V_h$. The eigenvalues $\{\lambda_{j,h}\}_{j=1}^{N_h}$ arranged in nondecreasing order:

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \ldots \leq \lambda_{N_h,h}.$$

with corresponding H-orthonormal eigenvectors $\mathcal{E} := \{e_{j,h}\}_{j=1}^{N_h}$ of L_h

Approximation: Galerkin finite elements

- $(V_h)_{h\in(0,1)}$ of subspaces of $\dot{H}^1$ with finite dimensions $N_h := \dim(V_h) < \infty$
- $\Pi_h \colon H \to V_h$ denotes the H-orthogonal projection onto V_h
- We denote version of L by $L_h \colon V_h \to V_h$, i.e., $(L_h\psi_h, \phi_h)_H = (L^{1/2}\psi_h, L^{1/2}\phi_h)_H$ for all $\psi_h, \phi_h \in V_h$. The eigenvalues $\{\lambda_{j,h}\}_{j=1}^{N_h}$ arranged in nondecreasing order:

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \ldots \leq \lambda_{N_h,h}.$$

with corresponding H-orthonormal eigenvectors $\mathcal{E} := \{e_{j,h}\}_{j=1}^{N_h}$ of L_h

Further assumptions:

- there exists a $d \in \mathbb{N}$ such that $N_h = \dim(V_h) \sim h^{-d}$ for all $h > 0$

Approximation: Galerkin finite elements

- $(V_h)_{h\in(0,1)}$ of subspaces of $\dot{H}^1$ with finite dimensions $N_h := \dim(V_h) < \infty$
- $\Pi_h\colon H \to V_h$ denotes the H-orthogonal projection onto V_h
- We denote version of L by $L_h\colon V_h \to V_h$, i.e., $(L_h\psi_h, \phi_h)_H = (L^{1/2}\psi_h, L^{1/2}\phi_h)_H$ for all $\psi_h, \phi_h \in V_h$. The eigenvalues $\{\lambda_{j,h}\}_{j=1}^{N_h}$ arranged in nondecreasing order:

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \ldots \leq \lambda_{N_h,h}.$$

  with corresponding H-orthonormal eigenvectors $\mathcal{E} := \{e_{j,h}\}_{j=1}^{N_h}$ of L_h

Further assumptions:

- there exists a $d \in \mathbb{N}$ such that $N_h = \dim(V_h) \sim h^{-d}$ for all $h > 0$
- there exist an $h_0 \in (0,1)$, constants $C_1, C_2 > 0$, as well as exponents $r, s > 0$, and $t > 1$ such that $\{\lambda_{j,h}\}$ and $\{e_{j,h}\}$ satisfy

$$\lambda_j \leq \lambda_{j,h} \leq \lambda_j + C_1 h^r \lambda_j^t,$$
$$\|e_j - e_{j,h}\|_H^2 \leq C_2 h^{2s} \lambda_j^t$$

  for all $h \in (0, h_0)$ and $j \in \{1, \ldots, N_h\}$

Approximation: Galerkin finite elements

- $(V_h)_{h\in(0,1)}$ of subspaces of $\dot{H}^1$ with finite dimensions $N_h := \dim(V_h) < \infty$
- $\Pi_h \colon H \to V_h$ denotes the H-orthogonal projection onto V_h
- We denote version of L by $L_h \colon V_h \to V_h$, i.e., $(L_h\psi_h, \phi_h)_H = (L^{1/2}\psi_h, L^{1/2}\phi_h)_H$ for all $\psi_h, \phi_h \in V_h$. The eigenvalues $\{\lambda_{j,h}\}_{j=1}^{N_h}$ arranged in nondecreasing order:

  $$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \ldots \leq \lambda_{N_h,h}.$$

  with corresponding H-orthonormal eigenvectors $\mathcal{E} := \{e_{j,h}\}_{j=1}^{N_h}$ of L_h

Further assumptions:

- there exists a $d \in \mathbb{N}$ such that $N_h = \dim(V_h) \sim h^{-d}$ for all $h > 0$
- there exist an $h_0 \in (0,1)$, constants $C_1, C_2 > 0$, as well as exponents $r, s > 0$, and $t > 1$ such that $\{\lambda_{j,h}\}$ and $\{e_{j,h}\}$ satisfy

  $$\lambda_j \leq \lambda_{j,h} \leq \lambda_j + C_1 h^r \lambda_j^t,$$
  $$\|e_j - e_{j,h}\|_H^2 \leq C_2 h^{2s} \lambda_j^t$$

  for all $h \in (0, h_0)$ and $j \in \{1, \ldots, N_h\}$

Example: if $\mathcal{D} \subset \mathbb{R}^d$ is a bounded, convex, polygonal domain, V_h is the finite element space of continuous, piecewise linear functions, then we have for $L = \kappa^2 - \Delta$ with homogeneous Dirichlet boundary conditions that $r = s = t = 2$

Noise approximation

We introduce the following V_h-valued random variables:

1. an expansion with respect to the discrete eigenbasis $\mathcal{E}$:

$$\dot{\mathcal{W}}_h^{\mathcal{E}} := \sum_{j=1}^{N_h} \xi_j \, e_{j,h}, \tag{2}$$

   where $\boldsymbol{\xi} := \left(\xi_1, \ldots, \xi_{N_h}\right)^T$ is the vector of the first N_h independent standard normally distributed random variables in the expansion of $\dot{\mathcal{W}}$

Noise approximation

We introduce the following V_h-valued random variables:

1. an expansion with respect to the discrete eigenbasis $\mathcal{E}$:

$$\dot{\mathcal{W}}_h^{\mathcal{E}} := \sum_{j=1}^{N_h} \xi_j \, e_{j,h}, \tag{2}$$

   where $\boldsymbol{\xi} := \left(\xi_1, \ldots, \xi_{N_h}\right)^T$ is the vector of the first N_h independent standard normally distributed random variables in the expansion of $\dot{\mathcal{W}}$

2. an expansion with respect to any basis $\Phi := \{\phi_{j,h}\}_{j=1}^{N_h}$ of V_h:

$$\dot{\mathcal{W}}_h^{\Phi} := \sum_{j=1}^{N_h} \widetilde{\xi}_j \, \phi_{j,h}, \tag{3}$$

   where the random vector $\widetilde{\boldsymbol{\xi}} := (\widetilde{\xi}_1, \ldots, \widetilde{\xi}_{N_h})^T$ is given by

$$\widetilde{\boldsymbol{\xi}} = \mathbf{R}^{-1}\boldsymbol{\xi}, \qquad R_{ij} := (e_{i,h}, \phi_{j,h})_H, \quad 1 \leq i, j \leq N_h$$

Noise approximation

We introduce the following V_h-valued random variables:

1. an expansion with respect to the discrete eigenbasis $\mathcal{E}$:

$$\dot{\mathcal{W}}_h^{\mathcal{E}} := \sum_{j=1}^{N_h} \xi_j \, e_{j,h}, \tag{2}$$

   where $\boldsymbol{\xi} := \left(\xi_1, \dots, \xi_{N_h}\right)^T$ is the vector of the first N_h independent standard normally distributed random variables in the expansion of $\dot{\mathcal{W}}$

2. an expansion with respect to any basis $\Phi := \{\phi_{j,h}\}_{j=1}^{N_h}$ of V_h:

$$\dot{\mathcal{W}}_h^{\Phi} := \sum_{j=1}^{N_h} \widetilde{\xi}_j \, \phi_{j,h}, \tag{3}$$

   where the random vector $\widetilde{\boldsymbol{\xi}} := (\widetilde{\xi}_1, \dots, \widetilde{\xi}_{N_h})^T$ is given by

$$\widetilde{\boldsymbol{\xi}} = \mathbf{R}^{-1}\boldsymbol{\xi}, \qquad R_{ij} := (e_{i,h}, \phi_{j,h})_H, \quad 1 \leq i,j \leq N_h$$

3. The random vector $\widetilde{\boldsymbol{\xi}}$ is therefore Gaussian distributed with zero mean and covariance matrix $\mathbf{R}^{-1}(\mathbf{R}^{-1})^T = (\mathbf{R}^T\mathbf{R})^{-1} = \mathbf{M}^{-1}$, where $\mathbf{M} = ((\phi_{i,h}, \phi_{j,h})_H)_{i,j=1}^{N_h}$ is the mass matrix with respect to the basis Φ of the finite element space V_h

Noise approximation

We introduce the following V_h-valued random variables:

1. an expansion with respect to the discrete eigenbasis $\mathcal{E}$:

$$\dot{\mathcal{W}}_h^{\mathcal{E}} := \sum_{j=1}^{N_h} \xi_j \, e_{j,h}, \tag{2}$$

   where $\boldsymbol{\xi} := \left(\xi_1, \ldots, \xi_{N_h}\right)^T$ is the vector of the first N_h independent standard normally distributed random variables in the expansion of $\dot{\mathcal{W}}$

2. an expansion with respect to any basis $\Phi := \{\phi_{j,h}\}_{j=1}^{N_h}$ of V_h:

$$\dot{\mathcal{W}}_h^{\Phi} := \sum_{j=1}^{N_h} \widetilde{\xi}_j \, \phi_{j,h}, \tag{3}$$

   where the random vector $\widetilde{\boldsymbol{\xi}} := (\widetilde{\xi}_1, \ldots, \widetilde{\xi}_{N_h})^T$ is given by

$$\widetilde{\boldsymbol{\xi}} = \mathbf{R}^{-1}\boldsymbol{\xi}, \qquad R_{ij} := (e_{i,h}, \phi_{j,h})_H, \quad 1 \leq i,j \leq N_h$$

3. The random vector $\widetilde{\boldsymbol{\xi}}$ is therefore Gaussian distributed with zero mean and covariance matrix $\mathbf{R}^{-1}(\mathbf{R}^{-1})^T = (\mathbf{R}^T\mathbf{R})^{-1} = \mathbf{M}^{-1}$, where $\mathbf{M} = ((\phi_{i,h}, \phi_{j,h})_H)_{i,j=1}^{N_h}$ is the mass matrix with respect to the basis Φ of the finite element space V_h

Lemma

The noise approximations $\dot{\mathcal{W}}_h^{\mathcal{E}}$ and $\dot{\mathcal{W}}_h^{\Phi}$ in (2)–(3) are equivalent in $L_2(\Omega; H)$, i.e., $\|\dot{\mathcal{W}}_h^{\mathcal{E}} - \dot{\mathcal{W}}_h^{\Phi}\|_{L_2(\Omega;H)} = 0$.

Functional calculus and quadrature

The approximation of $L^{-\beta}$ is based on the representation

$$L^{-\beta} = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \lambda^{-\beta}(\lambda I + L)^{-1}\,\mathrm{d}\lambda$$
$$= \frac{2\sin(\pi\beta)}{\pi} \int_{-\infty}^\infty e^{2\beta y}(I + e^{2y}L)^{-1}\,\mathrm{d}y$$

Functional calculus and quadrature

The approximation of $L^{-\beta}$ is based on the representation

$$L^{-\beta} = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \lambda^{-\beta} (\lambda I + L)^{-1} \,\mathrm{d}\lambda$$
$$= \frac{2\sin(\pi\beta)}{\pi} \int_{-\infty}^\infty e^{2\beta y} (I + e^{2y} L)^{-1} \,\mathrm{d}y$$

- We choose an equidistant grid $\{y_\ell = \ell k : \ell \in \mathbb{Z}, -K^- \leq \ell \leq K^+\}$ with step size $k > 0$ for y and quadrature method proposed by Bonito and Pasciak (2015)

$$L_h^{-\beta} \approx Q_{h,k}^\beta := \frac{2k\sin(\pi\beta)}{\pi} \sum_{\ell=-K^-}^{K^+} e^{2\beta y_\ell} (I + e^{2y_\ell} L_h)^{-1},$$

with exponential convergence of order $\mathcal{O}(e^{-\pi^2/(2k)})$ to $L_h^{-\beta}$ with respect to the norm

$$\|T\|_{\mathcal{L}(V_h)} := \sup_{\phi_h \in V_h \setminus \{0\}} \frac{\|T\phi_h\|_H}{\|\phi_h\|_H}$$

on the space $\mathcal{L}(V_h) := \{T \colon V_h \to V_h \text{ linear}\}$ for the choice

$$K^- := \left\lceil \frac{\pi^2}{4\beta k^2} \right\rceil, \qquad K^+ := \left\lceil \frac{\pi^2}{4(1-\beta)k^2} \right\rceil$$

Main convergence results

Define the numerical approximation $u^Q_{h,k}$ of the solution u to (1) as

$$u^Q_{h,k} := Q^\beta_{h,k}(\Pi_h g + \dot{\mathcal{W}}^\Phi_h). \tag{4}$$

Main convergence results

Define the numerical approximation $u_{h,k}^Q$ of the solution u to (1) as

$$u_{h,k}^Q := Q_{h,k}^{\beta}(\Pi_h g + \dot{\mathcal{W}}_h^{\Phi}). \tag{4}$$

Theorem (Strong Convergence)

Assume that the growth exponent α of the eigenvalues of the operator L satisfies

$$\frac{1}{2\beta} < \alpha \leq \min\left\{\frac{r}{(t-1)d}, \frac{2s}{td}\right\}.$$

Then, for sufficiently small $h \in (0, h_0)$ and $k \in (0, k_0)$, the strong $L_2(\Omega; H)$ error between the solution u of (1) and the approximation $u_{h,k}^Q$ in (4) is bounded by

$$\|u - u_{h,k}^Q\|_{L_2(\Omega;H)} \leq C\left(h^{\min\{d(\alpha\beta-1/2),r,s\}} + e^{-\pi^2/(2k)}h^{-d/2}\right)(1 + \|g\|_H),$$

where the constant $C > 0$ is independent of h and k.

Main convergence results

Define the numerical approximation $u^Q_{h,k}$ of the solution u to (1) as

$$u^Q_{h,k} := Q^\beta_{h,k}(\Pi_h g + \dot{\mathcal{W}}^\Phi_h). \tag{4}$$

Theorem (Strong Convergence)

Assume that the growth exponent α of the eigenvalues of the operator L satisfies

$$\tfrac{1}{2\beta} < \alpha \le \min\left\{\tfrac{r}{(t-1)d}, \tfrac{2s}{td}\right\}.$$

Then, for sufficiently small $h \in (0, h_0)$ and $k \in (0, k_0)$, the strong $L_2(\Omega; H)$ error between the solution u of (1) and the approximation $u^Q_{h,k}$ in (4) is bounded by

$$\|u - u^Q_{h,k}\|_{L_2(\Omega;H)} \le C\left(h^{\min\{d(\alpha\beta-1/2),r,s\}} + e^{-\pi^2/(2k)} h^{-d/2}\right)(1 + \|g\|_H),$$

where the constant $C > 0$ is independent of h and k.

Corollary (Weak type convergence)

Let the assumptions of the Theorem be satisfied. Then, there exists a constant $C > 0$ independent of $h \in (0, h_0)$ and $k \in (0, k_0)$ such that the following weak type error estimate between the approximation $u^Q_{h,k}$ in (4) and the solution u to (1) holds:

$$\begin{aligned}\left|\|u\|^2_{L_2(\Omega;H)} - \|u^Q_{h,k}\|^2_{L_2(\Omega;H)}\right| &\le C\left(h^{\min\{d(2\alpha\beta-1),r\}} + e^{-\pi^2/(2k)} h^{-d}\right) \\ &\quad + C\left(h^{\min\{d(2\alpha\beta-1),r,s\}} + e^{-\pi^2/(2k)}\right)\|g\|^2_H.\end{aligned}$$

Calibration

	calibration	rate of convergence
strong error	$k \leq -\frac{\pi^2}{2d\alpha\beta\ln(h)}$	$\min\{d(\alpha\beta - 1/2), r, s\}$
weak type error	$k \leq -\frac{\pi^2}{4d\alpha\beta\ln(h)}$	$\begin{cases} \min\{d(2\alpha\beta - 1), r\} & \text{if } g = 0 \\ \min\{d(2\alpha\beta - 1), r, s\} & \text{otherwise} \end{cases}$

Remarks

- The quadrature involves only non-fractional resolvents of the discrete operator L_h. Thus, the corresponding numerical method is readily implementable and local.

Remarks

- The quadrature involves only non-fractional resolvents of the discrete operator L_h. Thus, the corresponding numerical method is readily implementable and local.
- The exact eigenvectors of the operator L are not needed for the implementation.

Remarks

- The quadrature involves only non-fractional resolvents of the discrete operator L_h. Thus, the corresponding numerical method is readily implementable and local.
- The exact eigenvectors of the operator L are not needed for the implementation.
- For the computation of the approximation $u_{h,k}^Q$ in (4) in practice, one has to sample from the load vector $\mathbf{b}$ with entries

$$b_i := (\Pi_h g + \dot{\mathcal{W}}_h^\Phi, \phi_{i,h})_H, \qquad 1 \leq i \leq N_h,$$

  where $\{\phi_{j,h}\}_{j=1}^{N_h}$ is a basis of the finite element space V_h. If the basis Φ in the noise approximation $\dot{\mathcal{W}}_h^\Phi$ is chosen as the same, since then $\mathbf{b} \sim \mathcal{N}(\mathbf{g}, \mathbf{M})$, where $g_i = (g, \phi_{i,h})_H$ and $\mathbf{M}$ is the mass matrix with respect to the basis Φ, which is usually sparse. Hence, samples of $\mathbf{b}$ can be generated from $\mathbf{b} = \mathbf{g} + \mathbf{\Sigma}\mathbf{z}$, where $\mathbf{z} \sim \mathcal{N}(0, \mathbf{I})$ and $\mathbf{\Sigma}$ is a Cholesky factor of $\mathbf{M} = \mathbf{\Sigma}\mathbf{\Sigma}^T$.

Remarks

- The quadrature involves only non-fractional resolvents of the discrete operator L_h. Thus, the corresponding numerical method is readily implementable and local.
- The exact eigenvectors of the operator L are not needed for the implementation.
- For the computation of the approximation $u_{h,k}^Q$ in (4) in practice, one has to sample from the load vector $\mathbf{b}$ with entries

$$b_i := (\Pi_h g + \dot{\mathcal{W}}_h^{\Phi}, \phi_{i,h})_H, \qquad 1 \leq i \leq N_h,$$

  where $\{\phi_{j,h}\}_{j=1}^{N_h}$ is a basis of the finite element space V_h. If the basis Φ in the noise approximation $\dot{\mathcal{W}}_h^{\Phi}$ is chosen as the same, since then $\mathbf{b} \sim \mathcal{N}(\mathbf{g}, \mathbf{M})$, where $g_i = (g, \phi_{i,h})_H$ and $\mathbf{M}$ is the mass matrix with respect to the basis Φ, which is usually sparse. Hence, samples of $\mathbf{b}$ can be generated from $\mathbf{b} = \mathbf{g} + \mathbf{\Sigma z}$, where $\mathbf{z} \sim \mathcal{N}(0, \mathbf{I})$ and $\mathbf{\Sigma}$ is a Cholesky factor of $\mathbf{M} = \mathbf{\Sigma\Sigma}^T$.
- We also derived proper weak error estimates; that is, rates for

$$\left|\mathbb{E}[\varphi(u)] - \mathbb{E}[\varphi(u_{h,k}^Q)]\right|$$

  where $\varphi \colon H \to \mathbb{R}$ is twice continuously Frechet differentiable with polynomially growing derivatives. The proof is a lot more involved then the one for the strong error and uses a well-chosen time dependent problem so that we may use a Kolmogorov's equation in the analysis: $u = Y(1)$ in distribution where

$$\mathrm{d}Y(t) = \mathrm{d}W^{\beta}(t), \quad t \in [0,1], \qquad Y(0) = L^{-\beta}g, \quad W^{\beta}(t) := \sum_{j \in \mathbb{N}} \lambda_j^{-\beta} B_j(t)\, e_j,$$

  with an analogous representation for $u_{h,k}^Q$.

Numerical example

Let $\mathcal{D} := (0,1)^d$ in $d = 1, 2, 3$ spatial dimensions, we consider the following problem:

$$\begin{aligned} (\kappa^2 - \Delta)^\beta u(\mathbf{x}) &= \dot{\mathcal{W}}(\mathbf{x}) \qquad && \mathbf{x} \in \mathcal{D}, \\ u(\mathbf{x}) &= 0 && \mathbf{x} \in \partial\mathcal{D}, \end{aligned}$$

with $\kappa = 0.5$.

Numerical example

Let $\mathcal{D} := (0,1)^d$ in $d = 1, 2, 3$ spatial dimensions, we consider the following problem:

$$\begin{aligned} (\kappa^2 - \Delta)^\beta u(\mathbf{x}) &= \dot{\mathcal{W}}(\mathbf{x}) && \mathbf{x} \in \mathcal{D}, \\ u(\mathbf{x}) &= 0 && \mathbf{x} \in \partial\mathcal{D}, \end{aligned}$$

with $\kappa = 0.5$.

- We use a finite element discretization on a uniform mesh with continuous, piecewise linear basis functions

Numerical example

Let $\mathcal{D} := (0,1)^d$ in $d = 1,2,3$ spatial dimensions, we consider the following problem:

$$\begin{aligned} (\kappa^2 - \Delta)^\beta u(\mathbf{x}) &= \dot{\mathcal{W}}(\mathbf{x}) \qquad && \mathbf{x} \in \mathcal{D}, \\ u(\mathbf{x}) &= 0 && \mathbf{x} \in \partial\mathcal{D}, \end{aligned}$$

with $\kappa = 0.5$.

- We use a finite element discretization on a uniform mesh with continuous, piecewise linear basis functions
- Theory says: $\beta > d/4$ and rates $2\beta - d/2$ for the strong error and $\min\{4\beta - d, 2\}$ for the weak type error (after calibration of h and k)

Numerical example

Let $\mathcal{D} := (0,1)^d$ in $d = 1,2,3$ spatial dimensions, we consider the following problem:

$$\begin{aligned} (\kappa^2 - \Delta)^\beta u(\mathbf{x}) &= \dot{\mathcal{W}}(\mathbf{x}) && \mathbf{x} \in \mathcal{D}, \\ u(\mathbf{x}) &= 0 && \mathbf{x} \in \partial\mathcal{D}, \end{aligned}$$

with $\kappa = 0.5$.

- We use a finite element discretization on a uniform mesh with continuous, piecewise linear basis functions
- Theory says: $\beta > d/4$ and rates $2\beta - d/2$ for the strong error and $\min\{4\beta - d, 2\}$ for the weak type error (after calibration of h and k)
- Observed (resp. theoretical) rates of convergence for the strong errors:

	β				
	3/8	4/8	5/8	6/8	7/8
$d = 1$	0.25 (0.25)	0.50 (0.5)	0.75 (0.75)	1.00 (1)	1.21 (1.25)
$d = 2$	-	-	0.29 (0.25)	0.51 (0.5)	0.74 (0.75)
$d = 3$	-	-	-	-	0.26 (0.25)

Weak type error