Randomized Numerical Schemes

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Joint work with Dr. Raphael Kruse (TU Berlin)[1]

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Randomized Numerical Schemes for ODEs

Consider numerical approximation of

$$\dot{u}(t) = f(t, u(t)), \quad t \in [0, T], \quad u(0) = u_0, \tag{ODE}$$

where $f \colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is γ-Hölder continuous in t, $0 < \gamma \leq 1$; Lipschitz in state variable.

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Strategy: Deterministic Method '$\times$' Stratified Monte-Carlo

Stratified Monte-Carlo Integration

$(\tau_j)_j$: a seq. of $\mathcal{U}(0,1)$-distributed IID r.v. on $(\Omega_\tau, \mathcal{F}^\tau, \mathbb{P}_\tau)$.
To estimate a targeted integral:

$$\begin{aligned}\int_a^b z(t)dt =& (b-a)\int_a^b z(t)\frac{1}{(b-a)}dt \\ =& (b-a)\mathbb{E}_\tau\left(z(a+\tau(b-a))\right) \\ \overset{\text{Classic}}{\approx} & (b-a)\frac{1}{N}\sum_{i=1}^{N} z\left(a+\tau_i(b-a)\right) \\ \overset{\text{Stratified}}{\approx} & (b-a)\frac{1}{N}\sum_{i=0}^{N-1} z\left(a+\frac{(i+\tau_{i+1})}{N}(b-a)\right).\end{aligned}$$

Remark

Monte-Carlo convergence rate: $\mathcal{O}(N^{-\frac{1}{2}})$ *if* $z \in L^2$.

Randomized Quadrature Rule

- $(\tau_j)_j$: a seq. of $\mathcal{U}(0,1)$-distributed IID r.v. on $(\Omega_\tau, \mathcal{F}^\tau, \mathbb{P}_\tau)$.
- $z \colon [0,T] \to \mathbb{R}^d$, measurable, in $L^p([0,T];\mathbb{R}^d)$ with $p \in [2,\infty)$.

Define *randomized Riemann sum approx.* $Q^n_{\tau,h}[z]$ of $\int_0^{t_n} z(s)ds$

$$Q^n_{\tau,k}[z] := k \sum_{j=1}^{n} z(t_{j-1} + k\tau_j), \quad n \in \{1,\dots,N_k\}.$$

Theorem (Kruse&W., 2017a)

$Q^n_{\tau,k}[z] \in L^p(\Omega_\tau;\mathbb{R}^d)$ is an unbiased estimator. For all $k \in (0,1)$,

$$\Big\| \max_{n\in\{1,\dots,N_k\}} \Big| \int_0^{t_n} z(s)ds - Q^n_{\tau,k}[z] \Big| \Big\|_{L^p(\Omega_\tau;\mathbb{R})} \le C \|z\|_{\mathcal{C}^\gamma([0,T])} k^{\frac{1}{2}+\gamma}.$$

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Randomized Runge-Kutta method:

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for all $j \in \{1, 2, \cdots, N_k\}$.

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Remark (Stengle, 1990,1995; Jentzen&Neuenkirch, 2009)

$$U_j = U_{j-1} + k f(t_{j-1} + \tau_j k, U_{j-1}). \qquad \text{(Randomized Euler)}$$

Convergences for Randomized RK

Theorem (L^p convergence (Kruse&W., 2017a))

For given $k \in (0,1)$, we have

$$\Big\| \max_{j \in \{0,1,\dots,N_k\}} |u(t_j) - U_j| \Big\|_{L^p(\Omega_\tau;\mathbb{R})} \leq C k^{\frac{1}{2}+\gamma}.$$

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Theorem (Almost-sure convergence (Kruse&W., 2017a))

Let $(k_m)_{m\in\mathbb{N}}$ be an arbitrary sequence of stepsizes with $\sum_m k_m < \infty$. For every $\epsilon \in (0, \frac{1}{2})$ there exists a random variable $m_\epsilon^u \colon \Omega_\tau \to \mathbb{N}$ and a measurable set $A_\epsilon^u \in \mathcal{F}$ with $\mathbb{P}(A_\epsilon^u) = 1$ such that for every $\omega \in A_\epsilon^u$ and $m \ge m_\epsilon^u(\omega)$ we have

$$\max_{n \in \{0,1,\dots,N_{k_m}\}} |u(t_n) - U_n^{k_m}(\omega)| \le k_m^{\frac{1}{2}+\gamma-\epsilon}.$$

Randomized Numerical Schemes for SDEs

Given $W\colon [0,T]\times\Omega_W \to \mathbb{R}^m$ on $(\Omega_W, \mathcal{F}^W, (\mathcal{F}^W_t)_{t\in[0,T]}, \mathbb{P}_W)$.
Consider Milstein method for a $\mathbb{R}^d$-valued SDE

$$\mathrm{d}X(t) = f(t, X(t))\,\mathrm{d}t + \sum_{r=1}^{m} g^r(t, X(t))\,\mathrm{d}W^r(t), \quad t\in[0,T],$$
$$X(0) = X_0. \tag{SDE}$$

Remark (Kloeden and Platen, 1992)

If $f \in C^{1,1}$ and $g \in C^{1,2}$, convergence order of Milstein method is $\mathcal{O}(N^{-1})$.

Drift-randomized Milstein Scheme

Given the product PS $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_W \times \Omega_\tau, \mathcal{F}^W \otimes \mathcal{F}^\tau, \mathbb{P}_W \times \mathbb{P}_\tau)$. Then the *drift-randomized Milstein method* is

$$
\begin{aligned}
X_k^{j,\tau} =& X_k^{j-1} + k\tau_j f\left(t_{j-1}, X_k^{j-1}\right) + \sum_{r=1}^{m} g^r\left(t_{j-1}, X_k^{j-1}\right) I_{(r)}^{t_{j-1}, t_{j-1}+\tau_j k}, \\
X_k^{j} =& X_k^{j-1} + kf(t_{j-1} + k\tau_j, X_k^{j,\tau}) + \sum_{r=1}^{m} g^r(t_{j-1}, X_k^{j-1}) I_{(r)}^{t_{j-1}, t_j} \\
&+ \sum_{r_1, r_2=1}^{m} g^{r_1, r_2}(t_{j-1}, X_k^{j-1}) I_{(r_1, r_2)}^{t_{j-1}, t_j},
\end{aligned}
\tag{RM}
$$

where $I_{(r)}^{s,t}$ represents increment of W^r, and $I_{(r_1,r_2)}^{s,t}$ the iterated integral and $g^{r_1,r_2}(t,x) := \frac{\partial g^{r_1}}{\partial x}(t,x) g^{r_2}(t,x)$, for $r_1, r_2 \in \{1, 2, \cdots, m\}$.

Convergence with Maximum Order

Assumption (Drift coefficient function)

For continuous $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, there exists $C_f \in (0, \infty)$ s.t.

$$|f(t, x_1) - f(t, x_2)| \le C_f |x_1 - x_2|, \ \forall t \in [0, T], x_1, x_2 \in \mathbb{R}^d;$$
$$|f(t_1, x) - f(t_2, x)| \le C_f (1 + |x|)|t_1 - t_2|^\gamma, \ \forall t_1, t_2 \in [0, T], x \in \mathbb{R}^d.$$

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Theorem (Kruse&W., 2017b)

Under Assumptions on f *and* g, *the drift-randomized Milstein method is convergent of order* $\min\left(\frac{1}{2}+\gamma, 1\right)$.

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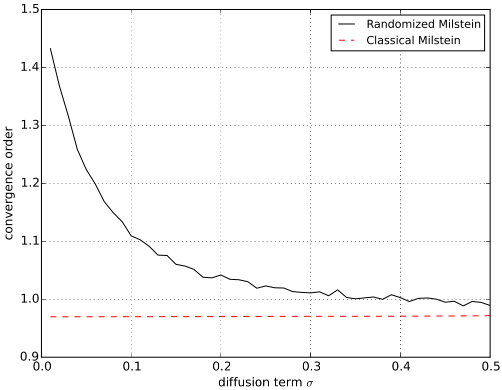
Remark

The maximum order of convergence for multiplicative case with numerical scheme (RM) is $\min\left(\frac{1}{2} + \gamma, 1\right)$.

Example 1

Consider the 1-dimensional GBM:

$$\mathrm{d}X(t) = X(t)dt + \sigma X(t)\,\mathrm{d}W(t), \quad t \in [0,1], \quad X(0) = 1.$$

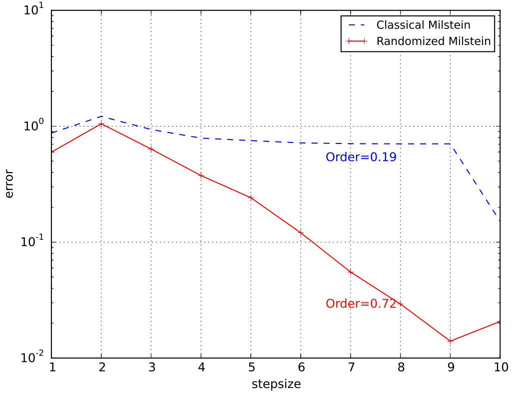


Example 2

Consider 1-dim SDE

$$\mathrm{d}X(t) = \Big(-\frac{|X(t)|}{1000} + |\sin(2^9\pi t)| \Big)\,\mathrm{d}t + |\cos(t)| X(t)\,\mathrm{d}W(t),$$

with $X(0) = 1.1$ and $t \in [0, 1]$.



Randomized Numerical Schemes for SPDEs

- $(H, (\cdot,\cdot), \|\cdot\|)$, $(U, (\cdot,\cdot)_U, \|\cdot\|_U)$: separable $\mathbb{R}$-Hilbert spaces;
- $(\Omega_W, \mathcal{F}^W, (\mathcal{F}^W_t)_{t\in[0,T]}, \mathbb{P}_W)$ a filtered probability space;
- $(W(t))_{t\in[0,T]}$: $(\mathcal{F}^W_t)_{t\in[0,T]}$-Wiener process on U with associated covariance operator $Q \in \mathcal{L}(U)$;
- $-A\colon \mathrm{dom}(A) \subset H \to H$, the infinitesimal generator of an analytic semigroup $(S(t))_{t\in[0,\infty)} \subset \mathcal{L}(H)$ on H.

Consider the mild solution to the H-valued non-autonomous semilinear stochastic evolution equation (SEE),

$$\begin{cases} \mathrm{d}X(t) + \left[AX(t) + f(t, X(t))\right]\mathrm{d}t = g(t)\,\mathrm{d}W(t), \\ X(0) = X_0, \quad \text{for } t \in (0, T]. \end{cases} \tag{SEE}$$

Fully-randomized Galerkin FEM

- $(V_h)_{h\in(0,1)} \subset H$, a suitable family of finite-dim subspaces;
- $h \in (0,1)$: such that $\lim_{h\to 0} \mathrm{dist}(u, V_h) = 0$ for all $u \in H$;
- $P_h \colon H \to V_h$, the orthogonal projectors onto V_h;
- $A_h \colon V_h \to V_h$, discrete version of the infinitesimal generator A.

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Then, the proposed *fully-randomized Galerkin FEM* is given by

$$
\begin{aligned}
X_{k,h}^{n,\tau} + \tau_n k\big[A_h X_{k,h}^{n,\tau} + P_h f(t_{n-1}, X_{k,h}^{n-1})\big] \\
= X_{k,h}^{n-1} + P_h g(t_{n-1}) \Delta_{\tau_n k} W(t_{n-1}), \\
X_{k,h}^{n} + k\big[A_h X_{k,h}^{n} + P_h f(t_{n-1} + \tau_n k, X_{k,h}^{n,\tau})\big] \\
= X_{k,h}^{n-1} + P_h g(t_{n-1} + \tau_n k) \Delta_k W(t_{n-1})
\end{aligned}
\tag{FRGalerkinFEM}
$$

for all $n \in \{1, \dots, N_k\}$ with initial value $X_{k,h}^0 = P_h X_0$.

Assumptions

Assumption (Drift)

For continous $f\colon [0,T]\times H \to H$, $\exists \gamma \in (0,1]$ *and* $C_f \in (0,\infty)$ *s.t.*

$$\|f(t,u_1) - f(t,u_2)\| \le C_f\|u_1 - u_2\|,$$
$$\|f(t_1,u) - f(t_2,u)\| \le C_f\big(1 + \|u\|\big)|t_1 - t_2|^\gamma$$

for all $u, u_1, u_2 \in H$ *and* $t, t_1, t_2 \in [0,T]$.

Assumption (Diffusion)

$g\colon [0,T] \to \mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}(U), H)$ *is continuous. Moreover,* $\exists p \in [2,\infty)$, $r \in [0,1]$, $\gamma \in (0,1]$, *and* $C_g \in (0,\infty)$ *s.t.*

$$\|g\|_{C^{\frac{1}{2}}(0,T;\mathcal{L}_2^0)} + \|A^{\frac{r}{2}} g\|_{C(0,T;\mathcal{L}_2^0)} + \|g\|_{\mathcal{W}^{\min(1,\frac{1}{2}+\gamma),p}(0,T;\mathcal{L}_2^0)} \le C_g.$$

Convergence

Theorem (Kruse &W., 2018)

Let Assumptions be fulfilled for some $p \in [2, \infty)$, $r \in [0, 1)$, *and* $\gamma \in (0, 1]$. *Then there exists a constant* $C \in (0, \infty)$ *such that for every* $h \in (0, 1]$ *and* $k \in (0, T)$

$$\max_{n \in \{0,\dots,N_k\}} \left\| X(t_n) - X^n_{k,h} \right\|_{L^p(\Omega;H)} \leq C\Big(1 + \|X\|_{C(0,T;L^p(\Omega_W;\dot{H}^{1+r}))} + \|X\|_{C^{\frac{1}{2}}(0,T;L^p(\Omega_W;\dot{H}^r))}\Big) \times \left(h^{1+r} + k^{\frac{1}{2}+\min(\frac{r}{2},\gamma)}\right),$$

where X *denotes the mild solution to the stochastic evolution equation* (SEE) *and* $(X^n_{k,h})_{n \in \{0,\dots,N_k\}}$ *denotes the stochastic process generated by the randomized Galerkin finite element method* (FRGalerkinFEM).

One experiment

Consider the numerical solution of the scalar SPDE

$$\begin{cases} \mathrm{d}u(t) & = (\triangle u(t) + f(u,t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t), \\ u(0,x) & = (1-x)x, \quad u(t,0) = u(t,1) = 0, \end{cases} \tag{EX}$$

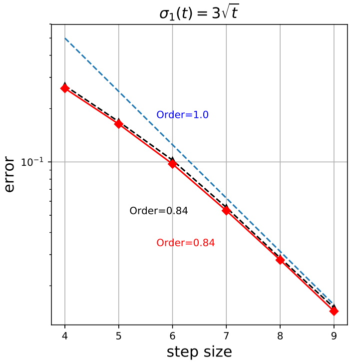
for $t \in [0,1]$ and $x \in [0,1]$. The drift function is given by

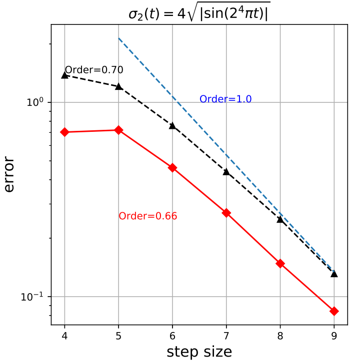
$$f(v,t) := \sum_{n=0}^{20} a^n \cos(b^n \pi v)$$

is a truncated version of Weierstrass function with $0 < a < 1$, b being odd and $ab > 1 + \frac{3}{2}\pi$. In our example, we chose $a = 0.9$ and $b = 7$ to ensure high fluctuation in v.

One experiment

The Python code is built on FEniCS package.





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THANK FOR YOUR ATTENTION!