Numerical analysis of a particle calibration procedure for local and stochastic volatility models

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Plan

1. Motivation
2. Weak error estimates
3. An interacting particle system
Processes matching marginal distributions

- Stochastic processes matching given marginals is a question arising in mathematical finance

- Assume that the market gives us the prices of European put options \( P(T, K) \) for all \( T, K \geq 0 \), on the underlying asset \( S \)

- Given a model on \( S \), the price of a put option is given by

\[
\mathbb{E}[e^{-rT}(K - S_T)_+] 
\]

- For hedging purposes, we want a model \( (S_t)_{t \geq 0} \) calibrated to those prices:

\[
\forall T, K \geq 0, \ P(T, K) = \mathbb{E} \left[ e^{-rT} (K - S_T)_+ \right] 
\]

- By Breeden and Litzenberger (1978), marginal laws are equivalent to market prices of European Puts \( P(T, K) \)
From LV to LSV

• Dupire calibrated Local Volatility model (1992) achieves exact calibration:
  \[ dS_t^D = rS_t^D dt + \sigma_{Dup}(t, S_t^D) S_t^D dW_t \]

• **Motivation**: richer dynamics but still satisfying marginal constraints

• Lipton (2002) and Piterbarg (2006): Local and Stochastic Volatility (LSV) model
  \[ dS_t = rS_t dt + f(Y_t) \sigma(t, S_t) S_t dW_t \]

• Stochastic volatility factor \( f \) fixed, choice of calibration function \( \sigma \) ?
Gyongy's Theorem

Let $X$ be an Ito process satisfying

$$dX_t = \alpha(t, \omega) \, dt + \beta(t, \omega) \, dW_t$$

where $\alpha, \beta$ are adapted processes. Under mild assumptions, there exists a Markov process $X^D_t$ satisfying

$$dX^D_t = a(t, X^D_t) \, dt + b(t, X^D_t) \, dW_t$$

where $X_t, X^D_t$ have the same distribution for all $t \geq 0$ and $X^D$ can be constructed with

$$a(t, y) = \mathbb{E}[\alpha(t, \omega) | X_t = y]$$

$$b^2(t, y) = \mathbb{E}[\beta^2(t, \omega) | X_t = y]$$
Calibration of LSV Models

- The LSV model is calibrated to \( (P(T, K))_{T,K \geq 0} \) if

\[
\mathbb{E} \left[ (f(Y_t)\sigma(t, S_t)S_t)^2 | S_t = x \right] = (\sigma_{Dup}(t, x)x)^2
\]

\[
\sigma(t, x) = \frac{\sigma_{Dup}(t, x)}{\sqrt{\mathbb{E} \left[ f^2(Y_t) | S_t = x \right]}}
\]

- The obtained SDE

\[
dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E} \left[ f^2(Y_t) | S_t \right]}} \sigma_{Dup}(t, S_t)S_t dW_t.
\]

is nonlinear in the sense of McKean and its solution should have the same one dimensional marginals as

\[
dS_t^D = rS_t^D dt + \sigma_{Dup}(t, S_t^D)S_t^D dW_t
\]

- **Questions**: existence and uniqueness? simulation?
Existence and uniqueness?

- Abergel, Tachet, 2010: local in time existence, perturbation of the Dupire model
- Jourdain, Z., 2017: global existence when $Y$ is a jump process with a finite number of states
- Existence and uniqueness to the SDE in the general setting remain open problems...
Simulation of the SDE

- Ren, Madan, Qian 2007: solve the associated Fokker-Planck PDE
- Guyon, Henry-Labordère 2008: kernel approximation of the conditional expectation and interacting particle system:

\[ X = \log(S), \tau_t = \left\lfloor \frac{nt}{T} \right\rfloor \frac{T}{n}, \text{ for } 1 \leq i \leq N, \]

\[
E_i \left[ f^2(Y_{\tau_t}^{n,i,N}) | X_{\tau_t}^{n,i,N} \right] = \frac{1}{N} \sum_{j=1}^{N} f^2(Y_{\tau_t}^{n,j,N}) K_{\epsilon} (X_{\tau_t}^{n,j,N} - X_{\tau_t}^{n,i,N}),
\]

\[
dX_{t}^{n,i,N} = \left( r - \frac{1}{2} E_i \left[ f^2(Y_{\tau_t}^{n,i,N}) | X_{\tau_t}^{n,i,N} \right] \sigma_{Dup}(\tau_t, X_{\tau_t}^{n,i,N}) \right) dt
\]

\[
+ \frac{f(Y_{\tau_t}^{n,i,N})}{\sqrt{E_i \left[ f^2(Y_{\tau_t}^{n,i,N}) | X_{\tau_t}^{n,i,N} \right]}} \sigma_{Dup}(\tau_t, X_{\tau_t}^{n,i,N}) dW_t
\]
Time discretization

The particle system is an efficient calibration procedure in the industry

Convergence and speed of calibration \( \frac{1}{N} \sum_{i=1}^{N} \varphi(X^{n,i,N}_T) \to E[\varphi(X^D_T)] \) ?

Step 1: Existence for calibrated LSV models seems challenging in the general case, but it is not a problem for its discretization in time. **General Framework:**
(multidimensional setting)

\[
\begin{align*}
\text{d}X^n_t &= b_X(t, X^n_{tt}, Y^n_{tt}, E[\phi(X^n_{tt}, Y^n_{tt})|X^n_{tt}]) \text{ d}t + \sigma_X(t, X^n_{tt}, Y^n_{tt}, E[\phi(X^n_{tt}, Y^n_{tt})|X^n_{tt}]) \text{ d}W^1_t, \\
\text{d}Y^n_t &= b_Y(t, X^n_{tt}, Y^n_{tt}) \text{ d}t + \sigma_Y(t, X^n_{tt}, Y^n_{tt}) \text{ d}W^2_t, \\
\text{d}X^D_t &= b(t, X^D_t) \text{ d}t + \sigma(t, X^D_t) \text{ d}W^1_t,
\end{align*}
\]

with the structure condition

\[
\begin{align*}
E[b_X(t, X^n_{tt}, Y^n_{tt}, E[\phi(X^n_{tt}, Y^n_{tt})|X^n_{tt}])|X^n_{tt}] &= b(t, X^n_{tt}) := \left( r - \frac{1}{2} \sigma_{Dup}^2(t, X^n_{tt}) \right) \\
E[\sigma_X^2(t, X^n_{tt}, Y^n_{tt}, E[\phi(X^n_{tt}, Y^n_{tt})|X^n_{tt}])|X^n_{tt}] &= \sigma^2(t, X^n_{tt}) := \sigma_{Dup}^2(t, X^n_{tt})
\end{align*}
\]
Weak Error estimates, regular case

- Multidimensional setting
- $b, \sigma \in C^{1,4}$ and bounded derivatives of positive order
- $\varphi \in C^4_P$
- $\phi, \sigma_X, \sigma_Y, b_X, b_Y$ sublinear w.r.t. their arguments
- $X_0$ and $Y_0$ have finite moments for all orders

Theorem (Regular case)

Under the above regularity conditions, there exists $C > 0$ such that

$$\forall n \geq 1, |\mathbb{E}[\varphi(X^n_T) - \varphi(X^D_T)]| \leq \frac{C}{n}.$$
Motivation

Weak error estimates

An interacting particle system

Sketch of proof 1/2

• All dimensions equal to 1
• \( b = 0, b_X = 0, \sigma = 1, \sigma_X \) bounded and \( \varphi \) smooth
• To study the weak error:
  \[
  \| \mathbb{E}[\varphi(X^n_T) - \varphi(X^D_T)] \|,
  \]
  
  let \( u(t, x) = \mathbb{E}[\varphi(X^D_T) | X^D_t = x] \). The function \( u \) is smooth and satisfies:
• \( \partial_t u + \frac{1}{2} \partial^2_{xx} u = 0, \ u(T, \cdot) = \varphi \) (heat equation)
• \( \mathbb{E}[\varphi(X^D_T)] = \mathbb{E}[u(T, X^D_T)] = \mathbb{E}[u(0, X^D_0)] \)

• Talay Tubaro technique:

  \[
  \mathbb{E}[\varphi(X^n_T) - \varphi(X^D_T)] = \sum_{k=0}^{n-1} \mathbb{E}[u(t_{k+1}, X^n_{t_{k+1}}) - u(t_k, X^n_{t_k})] = \sum_{k=0}^{n-1} E_k
  \]
Sketch of proof 2/2

- Notation \( \sigma_X \left( t_k, X_{t_k}^n, Y_{t_k}^n, \mathbb{E} \left[ \phi(X_{t_k}^n, Y_{t_k}^n) | X_{t_k}^n \right] \right) = \sigma_{X,k} \) (sim. for \( b_{X,k} \)) and recall the struct. cond.

\[
\forall 0 \leq k \leq n - 1, \mathbb{E}[\sigma_{X,k}^2 | X_{t_k}^n] = \sigma^2(t_k, X_{t_k}^n)
\]

- To study \( E_k \), we apply the Ito formula between \( t_k \) and \( t_k+1 \):

\[
u(t_{k+1}, X_{t_{k+1}}^n) - \nu(t_k, X_{t_k}^n) = \int_{t_k}^{t_{k+1}} \partial_x \nu(t, X_{t}^n) \sigma_{X,k} dW_t
\]

\[
+ \int_{t_k}^{t_{k+1}} \frac{1}{2} \left( \sigma_{X,k}^2 - \sigma^2(t_k, X_{t_k}^n) \right) \partial_x^2 \nu(t, X_{t}^n) dt
\]

\[
+ \int_{t_k}^{t_{k+1}} \frac{1}{2} \left( \sigma_{X}^2(t_k, X_{t_k}^n) - \sigma^2(t, X_{t}^n) \right) \partial_x^2 \nu(t, X_{t}^n) dt
\]

- Taylor expansion at order 2 to eliminate the lowest order:

\[
\partial_x^2 \nu(t, X_{t}^n) = \partial_x^2 \nu(t, X_{t_k}^n) + (X_{t}^n - X_{t_k}^n) \partial_x^3 \nu(t, X_{t_k}^n) + (X_{t}^n - X_{t_k}^n)^2 \mathcal{R}_k
\]

- Take the expectation and estimate the remaining terms

- We finally obtain that \( E_k \leq \frac{C}{n^2} \), so

\[
|\mathbb{E}[\phi(X_{T}^n) - \phi(X_{D}^T)]| \leq \frac{C}{n}
\]
Weak Error estimates, case of the put

- Unidimensional setting
- \( b, \sigma \in C^{1,\delta} \) and have bounded derivatives
- \( \phi, \sigma_Y, b_Y \) sublinear and \( b_X, \sigma_X \) bounded
- \( X_0 \) and \( Y_0 \) have finite moments for all orders

**Theorem (Case of the Put)**

*Under the above regularity conditions, for any \( K > 0 \), there exists \( C > 0 \) such that*

\[
\forall n \geq 2, \left| \mathbb{E}[ (K - e^{X^n_T})_+ - (K - e^{X^D_T})_+ ] \right| \leq C \frac{\log(n)}{n}.
\]

Same ideas of proof, estimates of the remainder terms are a bit different (gaussian estimates for the spatial derivatives of \( u \), Aronson estimates for the density of \( X^n_t \))
Half-step scheme

- Half step scheme, under uniform ellipticity $\sigma_X \geq \sigma > 0$

  \[
  \hat{X}_{t_{k+1/2}}^n = \hat{X}_{t_k}^n + \hat{b}_X, k\Delta + \sqrt{\hat{\sigma}_{X,k}^2 - \sigma^2} \sqrt{\Delta Z_k^1}
  \]

  \[
  \hat{X}_{t_{k+1}}^n = \hat{X}_{t_{k+1/2}}^n + \sigma \sqrt{\Delta Z_{k+1/2}^1}
  \]

  \[
  \hat{Y}_{t_{k+1}}^n = \hat{Y}_{t_k}^n + \hat{b}_Y, k\Delta + \hat{\sigma}_Y, k \sqrt{\Delta Z_k^2}
  \]

- For $\rho > 0, x \in \mathbb{R}, G_\rho(x) = \frac{1}{\sqrt{2\pi \rho}} e^{-\frac{x^2}{2\rho}}$

Proposition

For $k \geq 1$, $\hat{X}_{t_k}^n$ has the density $p^n_X(t_k, x) = \mathbb{E} \left[ G_{\sigma^2 \Delta} \left( x - \hat{X}_{t_k - \frac{1}{2}}^n \right) \right]$. Moreover, let

\[
\left( \tilde{X}_{t_k - \frac{1}{2}}^n, \tilde{Y}_{t_k}^n \right)
\]

be a copy of $\left( \hat{X}_{t_k - \frac{1}{2}}^n, \hat{Y}_{t_k}^n \right)$ independent of $\hat{X}_{t_k}^n$. The following representation holds:

\[
\mathbb{E} \left[ \phi \left( \hat{X}_{t_k}^n, \tilde{Y}_{t_k}^n \right) \mid \hat{X}_{t_k}^n \right] = \frac{\mathbb{E} \left[ \phi \left( \hat{X}_{t_k}^n, \hat{Y}_{t_k}^n \right) G_{\sigma^2 \Delta} \left( \hat{X}_{t_k}^n - \tilde{X}_{t_k - \frac{1}{2}}^n \right) \mid \hat{X}_{t_k}^n \right]}{\mathbb{E} \left[ G_{\sigma^2 \Delta} \left( \hat{X}_{t_k - \frac{1}{2}}^n - \tilde{X}_{t_k - \frac{1}{2}}^n \right) \mid \hat{X}_{t_k}^n \right]}
\]
Motivation

Weak error estimates

An interacting particle system

A particle system

For the particle system: replace the real law by the empirical law in the previous representation

\[
\begin{align*}
X_{t_{k+1/2}}^{n,i,N} &= X_{t_k}^{n,i,N} + b_{X,k}^{n,i,N} \Delta + \sqrt{(\sigma_{X,k}^{n,i,N})^2 - \sigma^2 \Delta Z_k^1}, \\
X_{t_{k+1}}^{n,i,N} &= X_{t_{k+1/2}}^{n,i,N} + \sigma \sqrt{\Delta Z_{k+1/2}^1}, \\
Y_{t_{k+1}}^{n,i,N} &= Y_{t_k}^{n,i,N} + b_Y(t_k, X_{t_k}^{n,i,N}, Y_{t_k}^{n,i,N}) \Delta + \sigma_Y(t_k, X_{t_k}^{n,i,N}, Y_{t_k}^{n,i,N}) \sqrt{\Delta Z_k^2}, \\
b_{X,k}^{n,i,N} &= b_X(t_k, X_{t_k}^{n,i,N}, Y_{t_k}^{n,i,N}, E_k^N(X_{t_k}^{n,i,N})) \\
\sigma_{X,k}^{n,i,N} &= \sigma_X(t_k, X_{t_k}^{n,i,N}, Y_{t_k}^{n,i,N}, E_k^N(X_{t_k}^{n,i,N})) \\
E_k^N(X_{t_k}^{n,i,N}) &= \frac{\sum_{j=1}^N \phi(X_{t_k}^{n,i,N}, Y_{t_k}^{n,j,N}) G_{\sigma^2 \Delta}(X_{t_k}^{n,i,N} - X_{t_k-\frac{1}{2}}^{n,j,N})}{\sum_{j=1}^N G_{\sigma^2 \Delta}(X_{t_k}^{n,i,N} - X_{t_k-\frac{1}{2}}^{n,j,N})}
\end{align*}
\]
Numerical illustration

- LSV Model with Black Scholes setting $\sigma = 1$, $r = 0$, $b = -\frac{1}{2}$, $f(y) = 1 + 1 \vee y^2$, $T = 1$, $Y_t = t + W_t$, atm put case $K = 1$, $X_0 = 0$

- Verification of time discretization weak error on the Kernel Approximation algorithm ($\epsilon = 0.1$, $N = 6000$)

<table>
<thead>
<tr>
<th>n</th>
<th>WE ($\times 10^{-4}$)</th>
<th>Product $\frac{n}{\log(n)}$ WE ($\times 10^{-4}$)</th>
</tr>
</thead>
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<tr>
<td>5</td>
<td>22.32</td>
<td>159.66</td>
</tr>
<tr>
<td>10</td>
<td>12.07</td>
<td>120.7</td>
</tr>
<tr>
<td>50</td>
<td>2.61</td>
<td>76.81</td>
</tr>
<tr>
<td>100</td>
<td>1.43</td>
<td>71.5</td>
</tr>
</tbody>
</table>
Numerical illustration

- $n = 5$ comparison of $MSE = \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \varphi(X_{T}^{n,i,N}) - \mathbb{E}[\varphi(X_{T}^{D})] \right)^2 \right]$ for the put $(K = 1, \text{atm})$ between optimized Kernel Approximation and Half Step scheme with maximal $\sigma = 0.25$ (no optimization)

- In the half-step scheme, the window of regularization is $\epsilon \sim \sqrt{\Delta}$, and as we expect $MSE \sim \Delta^2$, it would be consistent with the classical optimal NW rate $\epsilon_{opt}(N) \sim N^{-1/5}$

- Estimated exponent of MSE as function of particles $MSE \sim N^{-0.84}$

- Very roughly $\epsilon_{opt}(N) \sim N^{-0.24}$ but needs to be confirmed
Partial results on the convergence of the particle system to the calibrated one dimensional marginals:

- Convergence of the time discretized process at order 1 towards the calibrated marginal
- Half-step scheme taking advantage of a representation of the conditional expectation
- Relevant in our test case, to be confirmed with more simulations...
Thank you for your attention!