Numerical analysis of a particle calibration procedure for local and stochastic volatility models

Alexandre Zhou
Joint work with Benjamin Jourdain

CERMICS, Ecole des Ponts ParisTech

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Plan

Processes matching marginal distributions

- Stochastic processes matching given marginals is a question arising in mathematical finance
- Assume that the market gives us the prices of European put options $P(T, K)$ for all $T, K \geq 0$, on the underlying asset S
- Given a model on S, the price of a put option is given by

$$\mathbb{E}[e^{-rT}(K - S_T)_+]$$

- For hedging purposes, we want a model $(S_t)_{t\geq 0}$ calibrated to those prices:

$$\forall T, K \geq 0, \; P(T, K) = \mathbb{E}\left[e^{-rT}(K - S_T)_+\right]$$

- By Breeden and Litzenberger (1978), marginal laws are equivalent to market prices of European Puts $P(T, K)$

From LV to LSV

- Dupire calibrated Local Volatility model (1992) achieves exact calibration:

$$dS_t^D = rS_t^D dt + \sigma_{Dup}(t, S_t^D) S_t^D dW_t$$

- Motivation: richer dynamics but still satisfying marginal constraints

- Lipton (2002) and Piterbarg (2006): Local and Stochastic Volatility (LSV) model

$$dS_t = rS_t dt + f(Y_t)\sigma(t, S_t) S_t dW_t$$

- Stochastic volatility factor f fixed, choice of calibration function σ ?

Gyongy's Theorem

Let X be an Ito process satisfying

$$dX_t = \alpha(t,\omega)dt + \beta(t,\omega)dW_t$$

where α, β are adapted processes. Under mild assumptions, there exists a Markov process X_t^D satisfying

$$dX_t^D = a(t, X_t^D)dt + b(t, X_t^D)dW_t$$

where X_t, X_t^D have the same distribution for all $t \geq 0$ and X^D can be constructed with

$$\begin{aligned} a(t,y) =& \mathbb{E}[\alpha(t,\omega)|X_t = y] \\ b^2(t,y) =& \mathbb{E}[\beta^2(t,\omega)|X_t = y] \end{aligned}$$

Calibration of LSV Models

- The LSV model is calibrated to $(P(T,K))_{T,K\geq 0}$ if

$$\mathbb{E}\left[(f(Y_t)\sigma(t,S_t)S_t)^2|S_t = x\right] = (\sigma_{Dup}(t,x)x)^2$$

$$\sigma(t,x) = \frac{\sigma_{Dup}(t,x)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t = x\right]}}$$

- The obtained SDE

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t]}}\sigma_{Dup}(t,S_t)S_t dW_t.$$

is nonlinear in the sense of McKean and its solution should have the same one dimensional marginals as

$$dS_t^D = rS_t^D dt + \sigma_{Dup}(t,S_t^D)S_t^D dW_t$$

- **Questions**: existence and uniqueness? simulation?

Existence and uniqueness?

- Abergel, Tachet, 2010: local in time existence, perturbation of the Dupire model
- Jourdain, Z., 2017: global existence when Y is a jump process with a finite number of states
- Existence and uniqueness to the SDE in the general setting remain open problems...

Simulation of the SDE

- Ren, Madan, Qian 2007: solve the associated Fokker-Planck PDE
- Guyon, Henry-Labordère 2008: kernel approximation of the conditional expectation and interacting particle system:
- $X = \log(S), \tau_t = [\frac{nt}{T}]\frac{T}{n}$, for $1 \leq i \leq N$,

$$E_i\left[f^2(Y_{\tau_t}^{n,i,N})|X_{\tau_t}^{n,i,N}\right] = \frac{\frac{1}{N}\sum_{j=1}^{N} f^2(Y_{\tau_t}^{n,j,N})K_\epsilon(X_{\tau_t}^{n,j,N} - X_{\tau_t}^{n,i,N})}{\frac{1}{N}\sum_{i=1}^{N} K_\epsilon(X_{\tau_t}^{n,j,N} - X_{\tau_t}^{n,i,N})},$$

$$dX_t^{n,i,N} = \left(r - \frac{1}{2}\frac{f^2(Y_{\tau_t}^{n,i,N})}{E_i\left[f^2(Y_{\tau_t}^{n,i,N})|X_{\tau_t}^{n,i,N}\right]}\sigma_{Dup}(\tau_t, X_{\tau_t}^{n,i,N})\right)dt + \frac{f(Y_{\tau_t}^{n,i,N})}{\sqrt{E_i\left[f^2(Y_{\tau_t}^{i})|X_{\tau_t}^{n,i,N}\right]}}\sigma_{Dup}(\tau_t, X_{\tau_t}^{n,i,N})dW_t^i$$

Time discretization

The particle system is an efficient calibration procedure in the industry

Convergence and speed of calibration $\frac{1}{N}\sum_{i=1}^{N}\varphi(X_T^{n,i,N}) \underset{n,N\to\infty}{\to} \mathbb{E}[\varphi(X_T^D)]$?

Step 1: Existence for calibrated LSV models seems challenging in the general case, but it is not a problem for its discretization in time. **General Framework:** (multidimensional setting)

$$
\begin{aligned}
dX_t^n =& b_X\left(\tau_t, X_{\tau_t}^n, Y_{\tau_t}^n, \mathbb{E}\left[\phi(X_{\tau_t}^n, Y_{\tau_t}^n)|X_{\tau_t}^n\right]\right) dt + \sigma_X\left(\tau_t, X_{\tau_t}^n, Y_{\tau_t}^n, \mathbb{E}\left[\phi(X_{\tau_t}^n, Y_{\tau_t}^n)|X_{\tau_t}^n\right]\right) dW_t^1,\\
dY_t^n =& b_Y\left(\tau_t, X_{\tau_t}^n, Y_{\tau_t}^n\right) dt + \sigma_Y\left(\tau_t, X_{\tau_t}^n, Y_{\tau_t}^n\right) dW_t^2,\\
dX_t^D =& b\left(t, X_t^D\right) dt + \sigma\left(t, X_t^D\right) dW_t^1,
\end{aligned}
$$

with the structure condition

$$
\begin{aligned}
\mathbb{E}\left[b_X\left(\tau_t, X_{\tau_t}^n, Y_{\tau_t}^n, \mathbb{E}\left[\phi(X_{\tau_t}^n, Y_{\tau_t}^n)|X_{\tau_t}^n\right]\right)|X_{\tau_t}^n\right] =& b(\tau_t, X_{\tau_t}^n) := \left(r - \frac{1}{2}\sigma_{Dup}^2(\tau_t, X_{\tau_t}^n)\right)\\
\mathbb{E}\left[\sigma_X^2\left(\tau_t, X_{\tau_t}^n, Y_{\tau_t}^n, \mathbb{E}\left[\phi(X_{\tau_t}^n, Y_{\tau_t}^n)|X_{\tau_t}^n\right]\right)|X_{\tau_t}^n\right] =& \sigma^2(\tau_t, X_{\tau_t}^n) := \sigma_{Dup}^2(\tau_t, X_{\tau_t}^n)
\end{aligned}
$$

Weak Error estimates, regular case

- Multidimensional setting
- $b, \sigma \in C^{1,4}$ and bounded derivatives of positive order
- $\varphi \in C_P^4$
- ϕ, $\sigma_X, \sigma_Y, b_X, b_Y$ sublinear w.r.t. their arguments
- X_0 and Y_0 have finite moments for all orders

Theorem (Regular case)

Under the above regularity conditions, there exists $C > 0$ such that

$$\forall n \geq 1, |\mathbb{E}[\varphi(X_T^n) - \varphi(X_T^D)]| \leq \frac{C}{n}.$$

Sketch of proof 1/2

- All dimensions equal to 1
- $b = 0, b_X = 0, \sigma = 1, \sigma_X$ bounded and φ smooth
- To study the weak error:
$$|\mathbb{E}[\varphi(X_T^n) - \varphi(X_T^D)]|,$$
let $u(t,x) = \mathbb{E}[\varphi(X_T^D)|X_t^D = x]$. The function u is smooth and satisfies:
- $\partial_t u + \frac{1}{2}\partial_{xx}^2 u = 0,\ u(T,\cdot) = \varphi$ (heat equation)
- $\mathbb{E}[\varphi(X_T^D)] = \mathbb{E}[u(T, X_T^D)] = \mathbb{E}[u(0, X_0^D)]$
- Talay Tubaro technique:

$$\mathbb{E}[\varphi(X_T^n) - \varphi(X_T^D)] = \sum_{k=0}^{n-1} \mathbb{E}[u(t_{k+1}, X_{t_{k+1}}^n) - u(t_k, X_{t_k}^n)] = \sum_{k=0}^{n-1} E_k$$

Sketch of proof 2/2

- Notation $\sigma_X\left(t_k, X^n_{t_k}, Y^n_{t_k}, \mathbb{E}\left[\phi(X^n_{t_k}, Y^n_{t_k})|X^n_{t_k}\right]\right) = \sigma_{X,k}$ (sim. for $b_{X,k}$) and recall the struct. cond.

$$\forall 0 \leq k \leq n-1, \mathbb{E}[\sigma^2_{X,k}|X^n_{t_k}] = \sigma^2(t_k, X^n_{t_k})$$

- To study E_k, we apply the Ito formula between t_k and t_{k+1}:

$$\begin{aligned} u(t_{k+1}, X^n_{t_k+1}) - u(t_k, X^n_{t_k}) = &\int_{t_k}^{t_{k+1}} \partial_x u(t, X^n_t)\sigma_{X,k} dW_t \\ &+ \int_{t_k}^{t_{k+1}} \frac{1}{2}\left(\sigma^2_{X,k} - \sigma^2(t_k, X^n_{t_k})\right)\partial^2_x u(t, X^n_t) dt \\ &+ \int_{t_k}^{t_{k+1}} \frac{1}{2}\left(\sigma^2(t_k, X^n_{t_k}) - \sigma^2(t, X^n_t)\right)\partial^2_x u(t, X^n_t) dt \end{aligned}$$

- Taylor expansion at order 2 to eliminate the lowest order:

$$\partial^2_x u(t, X^n_t) = \partial^2_x u(t, X^n_{t_k}) + (X^n_t - X^n_{t_k})\partial^3_x u(t, X^n_{t_k}) + (X^n_t - X^n_{t_k})^2 \mathcal{R}_k$$

- Take the expectation and estimate the remaining terms
- We finally obtain that $E_k \leq \frac{C}{n^2}$, so

$$|\mathbb{E}[\varphi(X^n_T) - \varphi(X^D_T)]| \leq \frac{C}{n}$$

Weak Error estimates, case of the put

- Unidimensional setting
- $b, \sigma \in C^{1,6}$ and have bounded derivatives
- ϕ, σ_Y, b_Y sublinear and b_X, σ_X bounded
- X_0 and Y_0 have finite moments for all orders

Theorem (Case of the Put)

Under the above regularity conditions, for any $K > 0$, there exists $C > 0$ such that

$$\forall n \geq 2, |\mathbb{E}[(K - e^{X_T^n})_+ - (K - e^{X_T^D})_+]| \leq C\frac{\log(n)}{n}.$$

Same ideas of proof, estimates of the remainder terms are a bit different (gaussian estimates for the spatial derivatives of u, Aronson estimates for the density of X_t^n)

Half-step scheme

- Half step scheme, under uniform ellipticity $\sigma_X \geq \underline{\sigma} > 0$

$$\begin{aligned}
\hat{X}^n_{t_{k+1/2}} &= \hat{X}^n_{t_k} + \hat{b}_{X,k}\Delta + \sqrt{\hat{\sigma}^2_{X,k} - \underline{\sigma}^2}\sqrt{\Delta}Z^1_k \\
\hat{X}^n_{t_{k+1}} &= \hat{X}^n_{t_{k+1/2}} + \underline{\sigma}\sqrt{\Delta}Z^1_{k+1/2} \\
\hat{Y}^n_{t_{k+1}} &= \hat{Y}^n_{t_k} + \hat{b}_{Y,k}\Delta + \hat{\sigma}_{Y,k}\sqrt{\Delta}Z^2_k
\end{aligned}$$

- For $\rho > 0, x \in \mathbb{R}$, $G_\rho(x) = \frac{1}{\sqrt{2\pi\rho}} e^{\frac{-x^2}{2\rho}}$

Proposition

For $k \geq 1$, $\hat{X}^n_{t_k}$ has the density $p^n_X(t_k, x) = \mathbb{E}\left[G_{\underline{\sigma}^2\Delta}\left(x - \hat{X}^n_{t_{k-\frac{1}{2}}}\right)\right]$. Moreover, let $\left(\tilde{X}^n_{t_{k-\frac{1}{2}}}, \tilde{Y}^n_{t_k}\right)$ be a copy of $\left(\hat{X}^n_{t_{k-\frac{1}{2}}}, \hat{Y}^n_{t_k}\right)$ independent of $\hat{X}^n_{t_k}$. The following representation holds:

$$\mathbb{E}[\phi\left(\hat{X}^n_{t_k}, \hat{Y}^n_{t_k}\right)|\hat{X}^n_{t_k}] = \frac{\mathbb{E}\left[\phi\left(\hat{X}^n_{t_k}, \tilde{Y}^n_{t_k}\right) G_{\underline{\sigma}^2\Delta}\left(\hat{X}^n_{t_k} - \tilde{X}^n_{t_{k-\frac{1}{2}}}\right)|\hat{X}^n_{t_k}\right]}{\mathbb{E}\left[G_{\underline{\sigma}^2\Delta}\left(\hat{X}^n_{t_k} - \tilde{X}^n_{t_{k-\frac{1}{2}}}\right)|\hat{X}^n_{t_k}\right]}$$

A particle system

For the particle system: replace the real law by the empirical law in the previous representation

$$
\begin{aligned}
\mathbf{X}^{n,i,N}_{t_{k+1/2}} &= \mathbf{X}^{n,i,N}_{t_k} + b^{n,i,N}_{X,k}\Delta + \sqrt{(\sigma^{n,i,N}_{X,k})^2 - \underline{\sigma}^2}\sqrt{\Delta}Z^1_k,\\
\mathbf{X}^{n,i,N}_{t_{k+1}} &= \mathbf{X}^{n,i,N}_{t_{k+1/2}} + \underline{\sigma}\sqrt{\Delta}Z^1_{k+1/2},\\
\mathbf{Y}^{n,i,N}_{t_{k+1}} &= \mathbf{Y}^{n,i,N}_{t_k} + b_Y(t_k, \mathbf{X}^{n,i,N}_{t_k}, \mathbf{Y}^{n,i,N}_{t_k})\Delta + \sigma_Y(t_k, \mathbf{X}^{n,i,N}_{t_k}, \mathbf{Y}^{n,i,N}_{t_k})\sqrt{\Delta}Z^2_k,\\
b^{n,i,N}_{X,k} &= b_X(t_k, \mathbf{X}^{n,i,N}_{t_k}, \mathbf{Y}^{n,i,N}_{t_k}, E^N_k(\mathbf{X}^{n,i,N}_{t_k}))\\
\sigma^{n,i,N}_{X,k} &= \sigma_X(t_k, \mathbf{X}^{n,i,N}_{t_k}, \mathbf{Y}^{n,i,N}_{t_k}, E^N_k(\mathbf{X}^{n,i,N}_{t_k}))\\
E^N_k(\mathbf{X}^{n,i,N}_{t_k}) &= \frac{\sum_{j=1}^N \phi(\mathbf{X}^{n,i,N}_{t_k}, \mathbf{Y}^{n,j,N}_{t_k}) G_{\underline{\sigma}^2\Delta}(\mathbf{X}^{n,i,N}_{t_k} - \mathbf{X}^{n,j,N}_{t_k - \frac{1}{2}})}{\sum_{j=1}^N G_{\underline{\sigma}^2\Delta}(\mathbf{X}^{n,i,N}_{t_k} - \mathbf{X}^{n,j,N}_{t_k - \frac{1}{2}})}
\end{aligned}
$$

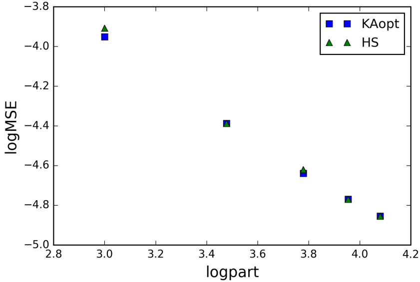
Numerical illustration

- LSV Model with Black Scholes setting $\sigma = 1,\ r = 0,\ b = -\frac{1}{2},\ f(y) = 1 + 1 \wedge y^2,$ $T = 1,\ Y_t = t + W_t,$ atm put case $K = 1, X_0 = 0$
- Verification of time discretization weak error on the Kernel Approximation algorithm ($\epsilon = 0.1, N = 6000$)

n	WE ($\times 10^{-4}$)	Product $\frac{n}{\log(n)} WE$ ($\times 10^{-4}$)
5	22.32	159.66
10	12.07	120.7
50	2.61	76.81
100	1.43	71.5

Numerical illustration

- $n = 5$ comparison of $MSE = \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\varphi(\mathbf{X}_T^{n,i,N}) - \mathbb{E}[\varphi(X_T^D)]\right)^2\right]$ for the put ($K = 1$, atm) between optimized Kernel Approximation and Half Step scheme with maximal $\underline{\sigma} = 0.25$ (no optimization)



- In the half-step scheme, the window of regularization is $\epsilon \sim \sqrt{\Delta}$, and as we expect $MSE \sim \Delta^2$, it would be consistent with the classical optimal NW rate $\epsilon_{opt}(N) \sim N^{-1/5}$
- Estimated exponent of MSE as function of particles $MSE \sim N^{-0.84}$
- very roughly $\epsilon_{opt}(N) \sim N^{-0.24}$ but needs to be confirmed

Conclusion

Partial results on the convergence of the particle system to the calibrated one dimensional marginals:

- Convergence of the time discretized process at order 1 towards the calibrated marginal
- Half-step scheme taking advantage of a representation of the conditional expectation
- Relevant in our test case, to be confirmed with more simulations...

Thank you!

Thank you for your attention!