MEAN REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS: Simulation by using Particle Systems

Ph. Briand[1] A. Ghannoum[1,2] C. Labart[1]

[1]LAMA, Savoie Mont Blanc University

[2]LAMA-Liban, Lebanese University

13th International Conference in Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing
Rennes, July 2018

Outline

Stochastic Differential Equations (SDE):

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) dB_s, \quad t \geq 0.$$

- X :Stochastic process.
- X_0: Random vector on $(\Omega, \mathcal{F})$.
- B: Brownian motion.
- $b, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ are two measurable and Lipschitz-continuous functions.

Preliminary results :

- Existence and uniqueness of the solution.
- Numerical solution based on Euler scheme.

Reflected SDE (RSDE):

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)dB_s + K_t, \quad 0 \le t \le T, \\ X_t \in \bar{D}, \quad K_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} n(X_s)\,d|K|_s. \end{cases}$$

- X is a process reflecting on the boundaries of domain $\bar{D}$.
- $b, \sigma : \bar{D} \longrightarrow \mathbb{R}$ are two Lipschitz-continuous functions.
- K is a bounded variations process with variation $|K|$.

Remark : The constraint acts on the paths of the process X.

Preliminary results :

- Existence and uniqueness of the solution (X, K).
- Numerical solution based on Euler scheme.

Mean RSDE (MRSDE):

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) dB_s + K_t, \quad t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)]\, dK_s = 0, \quad t \geq 0. \end{cases}$$

Main property :

- The constraint acts on the law of the process X rather than on its paths.
- Continuous case (EDS without jumps).

Preliminary results :

- Existence and uniqueness of the solution ([BEH18]).
- Numerical solution based on particles system and the convergence ([BCdRGL16]).

Objectives :

- Study of discontinuous case by adding:
  - Poisson Process (jumps=1).
  - Levy Process.

Methodology :

- Existence and uniqueness of solution.
- Particles system.
- Numerical scheme.
- Strong convergence.
- Numerical illustrations.

MRSDE with jumps

Extension of the previous results to the case of jumps :

$$\begin{cases} X_t = X_0 + \int_0^t b(X_{s^-})ds + \int_0^t \sigma(X_{s^-})dB_s \\ \qquad + \int_0^t F(X_{s^-})d\tilde{N}_s + K_t, \quad t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)]\, dK_s = 0, \quad t \geq 0. \end{cases}$$

- $\tilde{N}$: Compensated Poisson process ($\tilde{N}_t = N_t - \lambda t$).
- B: Brownian process independent of N.

Outline

Assumption (A.1) :

- $b:\mathbb{R}\longrightarrow\mathbb{R}$, $\sigma:\mathbb{R}\longrightarrow\mathbb{R}$ and $F:\mathbb{R}\longrightarrow\mathbb{R}$ are Lipschitz continuous functions.
- X_0 is square integrable independent of B_t and N_t.

Assumption (A.2) :

- $h:\mathbb{R}\longrightarrow\mathbb{R}$ is an increasing function and there exist $0<m\leq M$ such that

$$\forall x\in\mathbb{R},\ \forall y\in\mathbb{R},\ m|x-y|\leq|h(x)-h(y)|\leq M|x-y|.$$

- $\mathbb{E}[h(X_0)]\geq 0$.

Assumption (A.3) :

- $\exists p>4$ such that: $\mathbb{E}[|X_0|^p]<\infty$.

Assumption (A.4) :

- h is a twice continuously differentiable function with bounded derivatives.

Existence and uniqueness of the solution

Theorem

Under Assumptions (A.1) and (A.2), the MRSDE has a unique deterministic flat solution (X, K). Moreover,

$$\forall t \geq 0,\ K_t = \sup_{s \leq t} \inf\{y \geq 0 : \mathbb{E}[h(y + U_s)] \geq 0\}$$
$$= \sup_{s \leq t} G_0(\mu_s),$$

$$U_t = X_0 + \int_0^t b(X_{s^-})ds + \int_0^t \sigma(X_{s^-})dB_s + \int_0^t F(X_{s^-})d\tilde{N}_s.$$

- $G_0 : \mathcal{P}(\mathbb{R}) \ni \nu \mapsto \inf\{y \ \geq 0 : \int h(y+z)\nu(dz) \geq 0\}$.
- $(\mu_t)_{0 \leq t \leq T}$: Family of marginal laws of $(U_t)_{0 \leq t \leq T}$.

Outline

Solution of SDE:

$$X_t = X_0 + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s + \int_0^t F(X_{s-})d\tilde{N}_s + \sup_{s\le t} G_0(\mu_s).$$

$$\Downarrow$$

Particles approximation: $\forall i \in \{1, \cdots, N\}$,

$$X_t^i = \bar{X}_0^i + \int_0^t b(X_{s-}^i)ds + \int_0^t \sigma(X_{s-}^i)dB_s^i + \int_0^t F(X_{s-}^i)d\tilde{N}_s{}^i + \sup_{s\le t} G_0(\mu_s^N).$$

- B^i: independent Brownian motions.
- $\tilde{N}^i$: independent compensated Poisson processes.
- $\bar{X}_0^i$: independent copies of X_0.

μ_s^N denotes the empirical distribution at time s of the particles

$$U_t^i = \bar{X}_0^i + \int_0^t b(X_{s^-}^i)ds + \int_0^t \sigma(X_{s^-}^i)dB_s^i + \int_0^t F(X_{s^-}^i)d\tilde{N}_s{}^i,$$

namely

$$\mu_s^N = \frac{1}{N}\sum_{i=1}^N \delta_{U_s^i}.$$

Note that

$$G_0(\mu_s^N) = \inf\left\{x \geq 0 : \frac{1}{N}\sum_{i=1}^N h(x + U_s^i) \geq 0\right\}.$$

Propagation of chaos

Theorem

Let $T > 0$ and $N \in \mathbb{N}$. Under Assumptions (A.1), (A.2) and (A.3), there exists a constant $C_{(T,b,\sigma,F,h,X_0)}$, such that: $\forall i = 1, \dots, N$

(i)

$$\mathbb{E}\left[\sup_{s\leq T} |X_s^i - \bar{X}_s^i|^2\right] \leq CN^{-1/2}.$$

(ii) *Moreover, if $h \in C_b^2$,*

$$\mathbb{E}\left[\sup_{s\leq T} |X_s^i - \bar{X}_s^i|^2\right] \leq CN^{-1}.$$

- N: Number of particles.
- X_s^i: Particles system. $\bar{X}_s^i$: Independents copies of X_s.

Outline

Euler scheme applied to the particles system

Theorem

Let $T > 0$ and $N, n \in \mathbb{N}$. Under Assumptions (A.1), (A.2) and (A.3), $\exists$ constant $C_{(T,b,\sigma,F,h,X_0)}$, such that: $\forall i = 1, \ldots, N$

(i)

$$\mathbb{E}\left[\sup_{t\leq T}\left|\bar{X}_t^i - \tilde{X}_t^i\right|^2\right] \leq C\left(n^{-1} + N^{-1/2}\right).$$

(ii) *If in addition $h \in C_b^2$,*

$$\mathbb{E}\left[\sup_{t\leq T}\left|\bar{X}_t^i - \tilde{X}_t^i\right|^2\right] \leq C\left(n^{-1} + N^{-1}\right).$$

- $\bar{X}_t^i$: Independents copies of X (Exact solution).
- $\tilde{X}_t^i$: Numerical approximation (Euler scheme).
- n: Number of points of the discretization grid.

Outline

We approximate the $\mathbb{L}^2$-error by: For a given $i \in \{1, \cdots, N\}$,

$$\hat{E} = \left(\frac{1}{L} \sum_{l=1}^{L} \max_{0 \le k \le n} \left| \bar{X}^{l}_{T_k} - \tilde{X}^{i,l}_{T_k} \right|^2 \right)^{1/2}.$$

- $0 = T_0 < T_1 < \cdots < T_n = T$: a subdivision of $[0, T]$ of step size T/n.
- X: the unique solution of the MRSDE, $(\bar{X}^l)_{0 \le l \le L}$: L independent copies of X.
- $(\tilde{X}^i_{T_k})_{0 \le k \le n}$: a numerical approximation, $(\tilde{X}^{i,l})_{0 \le l \le L}$: L independent copies of $\tilde{X}^i$.

Linear constraint Let $h : \mathbb{R} \ni x \longmapsto x - p \in \mathbb{R}$.

1- Drifted Brownian motion and compensated Poisson process:

$$\begin{cases} X_t = x_0 - \beta \int_0^t ds + \sigma \int_0^t dB_s + \eta \int_0^t d\tilde{N}_s + K_t, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)] dK_s = 0, \quad t \geq 0. \end{cases}$$

We have

$$K_t = (p + \beta t - x_0)^+,$$

$$X_t = x_0 - (\beta + \eta\lambda)t + \sigma B_t + \eta N_t + K_t,$$

where $N_t \sim \mathcal{P}(\lambda t)$.

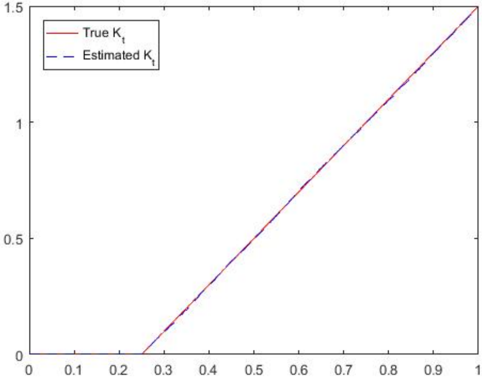

Figure: Case (1). Parameters: $n = 500$, $N = 100000$, $T = 1$, $\beta = 2$, $\sigma = 1$, $\lambda = 2$, $x_0 = 1$, $p = 1/2$.

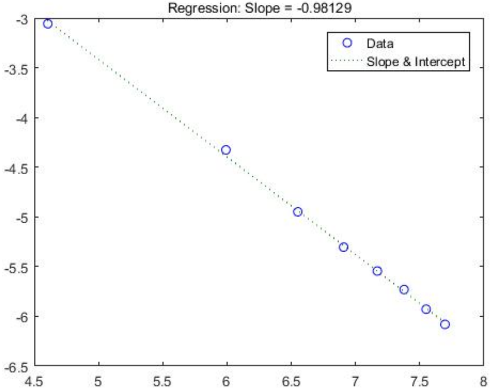

Figure: Case (1). Regression of $\log(\hat{E})$ w.r.t. $\log(N)$. Data: $\hat{E}$ when N varies from 100 to 2200 with step size 300. Parameters: $n = 100$, $T = 1$, $\beta = 2$, $\sigma = 1$, $\lambda = 2$, $x_0 = 1$, $p = 1/2$, $L = 1000$.

Nonlinear constraint Let $h : \mathbb{R} \ni x \longmapsto x + \alpha \sin(x) - p \in \mathbb{R}$, $-1 < \alpha < 1$.

2- Ornstein Uhlenbeck process:

$$\begin{cases} X_t = x_0 - \displaystyle\int_0^t (\beta + aX_{s^-})ds + \sigma \int_0^t dB_s + \eta \int_0^t d\tilde{N}_s \\ \qquad + K_t, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \displaystyle\int_0^t \mathbb{E}[h(X_s)]dK_s = 0, \quad t \geq 0. \end{cases}$$

We obtain

$$\mathrm{d}K_t = e^{-at}\mathrm{d}\sup_{s \leq t}(F_s^{-1}(0))^+,$$

where F_s is a deterministic function.

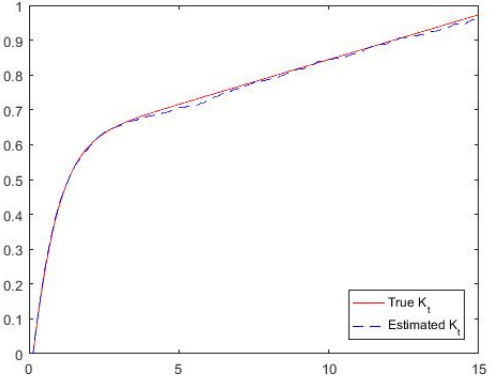

Figure: Case (3). Parameters: $n = 1000$, $N = 100000$, $T = 15$, $\beta = 10^{-2}$, $\sigma = 1$, $p = \pi/2$, $\alpha = 0.9$, $a = 10^{-2}$, x_0 is the unique solution of $x + \alpha \sin(x) - p = 0$ plus 10^{-1}.

Conclusion:

We can extend this result for the case of Levy process:

$$\begin{cases} X_t = X_0 + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s \\ \qquad + \int_0^t \int_E F(X_{s-}, z)\tilde{N}(ds, dz) + K_t, \quad t \geq 0, \\ \mathbb{E}[h(X_t)] \geq 0, \quad \int_0^t \mathbb{E}[h(X_s)]\, dK_s = 0, \quad t \geq 0. \end{cases}$$

- $E = \mathbb{R}^*$.
- Compensated Poisson measure, $\tilde{N}(ds, dz) = N(ds, dz) - \lambda(dz)ds$.

Thanks for your attention!

Appendix :

Proof of Theorem (existence and uniqueness) :

- We have

$$\mathbb{E}[h(U_t + K_t)] \geq 0 \;\Rightarrow\; K_t \geq \inf\{y \geq 0 : \mathbb{E}[h(y + U_t)] \geq 0\},$$

  and since K_t is an increasing process, we get

$$K_t \geq \sup_{s \leq t} \inf\{y \geq 0 : \mathbb{E}[h(y + U_s)] \geq 0\}.$$

- Under the bi-Lipschitz Assumption of h, we have

$$\forall \nu, \nu' \in \mathcal{P}(\mathbb{R}), \quad |G_0(\nu) - G_0(\nu')| \leq \frac{M}{m} W_1(\nu, \nu').$$

  Then, we use the fixed point theorem to conclude the results.

Note that W_1 is the Wasserstein-1 distance: $\forall \nu, \nu' \in \mathcal{P}(\mathbb{R})$,

$$W_1(\nu, \nu') = \sup_{\varphi \; 1-Lipschitz} \left| \int \varphi(d\nu - d\nu') \right| = \inf_{X \sim \nu \; ; \; Y \sim \nu'} \mathbb{E}[|X - Y|].$$

Proof of Theorem (convergence of particles system):
Using some majoration, we obtain

$$\mathbb{E}\Big[\sup_{s\le t}\Big|G_0(\bar{\mu}_s^N)-G_0(\mu_s)\Big|^2\Big] \le C\mathbb{E}\left[\sup_{s\le t}\left|\int h(\bar{G}_0(\mu_s)+\cdot)(d\bar{\mu}_s^N - d\mu_s)\right|^2\right].$$

Proof of (i): Under the bi-Lipschitz assumption of h and by adapting the proof of a theorem 10.2.7 of [RR98], we get

$$\le C\mathbb{E}\left[\sup_{s\le t} W_1^2(\bar{\mu}_s^N,\mu_s)\right] \le CN^{-1/2}.$$

Proof of (ii): (case: $h\in C_b^2$) Applying the Itô's formula, we get

$$\mathbb{E}\left[\sup_{s\le t}\left|\int h(\bar{G}_0(\mu_s)+\cdot)(d\bar{\mu}_s^N - d\mu_s)\right|^2\right] \le CN^{-1}.$$