On discrepancy and pair correlation of sequences in the unit interval

G. Larcher, JKU Linz

Devoted to the 80th birthday of Henri Faure

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- How to generate sequences $(x_n)_{n\geq 1}$ in $[0,1)$ with small discrepancy D_N
- How to generate point sets $(x_n)_{n=1,\ldots,N}$ in $[0,1)^2$ with small discrepancy
- at least for $N \to \infty$
  for N large

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$\exists\ \tilde{c}, \tilde{c}^*, \tilde{c}_p > 0$ such that for all $(x_n)_{n\geq 1}$ in $[0,1)$

$$D_N > \tilde{c} \cdot \frac{\log N}{N} \quad \text{for } \infty \text{ many } N$$

$$D_N^* > \tilde{c}^* \cdot \frac{\log N}{N} \quad \text{for } \infty \text{ many } N$$

W.M. Schmidt, 1972

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W.M. Schmidt, 1972

$$L_N^{(p)} > \tilde{c}_p \cdot \frac{\sqrt{\log N}}{N} \quad \text{for } \infty \text{ many } N$$

K.F. Roth, 1954	for $2 \leq p < \infty$
W.M. Schmidt, 1977	for $1 < p < 2$
G. Halasz, 1981	for $p = 1$

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symmetrized van der Corput-sequence $(x_n)_{n\geq 1}$:

$$L_N^{(p)} < \bar{c}_p \cdot \frac{\sqrt{\log N}}{N} \quad \underset{\text{all } 1\leq p<\infty}{\text{for all } N}$$

R. Kritzinger and F. Pillichshammer, 2015

Hence makes sense to define the (one-dimensional) discrepancy-constants:

$$c_\infty := \inf_w \limsup_{N\to\infty} \frac{N \cdot D_N(w)}{\log N}$$

$$c^* := \inf_w \limsup_{N\to\infty} \frac{N \cdot D_N^*(w)}{\log N}$$

$$c_p := \inf_w \limsup_{N\to\infty} \frac{N \cdot L_N^{(p)}(w)}{\sqrt{\log N}}$$

w sequence in $[0, 1)$.

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What is the "best distributed" sequence $(x_n)_{n\geq 1}$ in $[0,1)$ (with respect to $D_N, D_N^*, L_N^{(p)}$)?

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What is the "best distributed" sequence $(x_n)_{n\geq 1}$ in $[0, 1)$ (with respect to $D_N, D_N^*, L_N^{(p)}$)?

$\Rightarrow$ estimates for c_∞, c^*, c_p

$0.06015\ldots < c^* < 0.222223\ldots$

Bejian, 1982 Ostromoukhov, 2008

slightly improving

Faure, 1992

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$0.0515599 \; < c_2 < \; 0.3209\ldots$

Hinrichs, L., 2015 Faure, 1990

$0.1203\ldots < c_\infty < 0.353494$

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improved to

$0.1211\ldots$

L., 2017

Some notes:

Bejian
$0.1203\ldots < c_\infty$
$\Downarrow$
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$0.06015\ldots < c^*$
(since $D_N \leq 2 \cdot D_N^*$ always)

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$0.1211\ldots < c_\infty$ even harder!

Some notes:

$$0.06566\ldots < c^* < 0.222223\ldots$$

$$\downarrow$$

$$\exists\,(x_n)_{n\geq 1} \quad s.th.$$

$$D_N^* \leq 0.222224\ldots \cdot \frac{\log N}{N}$$

for all N large enough!

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for all N large enough!

indeed: holds for N **very** large only

$$D_N^* \leq 0.22224\ldots \cdot \frac{\log N}{N}$$

for all $N \geq N_0$

$\Rightarrow$

$$\exists\,(x_1, y_1), \ldots, (x_{N_0}, y_{N_0}) \in [0, 1)^2$$

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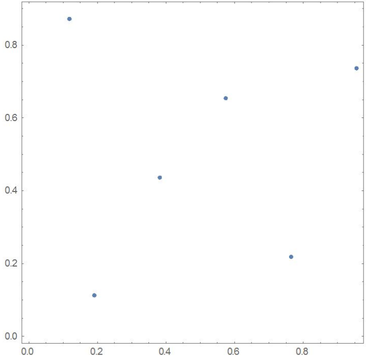
Consider "best" point sets $(x_1, y_1), \ldots, (x_{N_0}, y_{N_0}) \in [0, 1)^2$ with respect to $D_{N_0}^*$:

$$D^*_{N_0} \leq 0.22224 \cdot \frac{\log N_0}{N_0} \;?$$

e.g.: $N_0 = 6$, White 1975

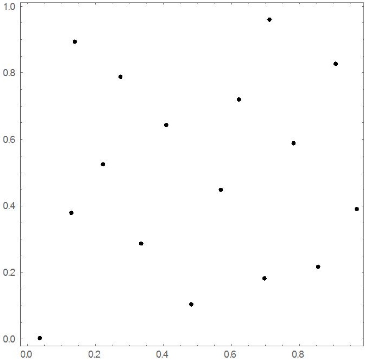


best configuration at all

$$D^*_6 = 0.5581 \cdot \frac{\log 6}{6}$$

$$D^*_{N_0} \leq 0.22224 \cdot \frac{\log N_0}{N_0} \;?$$

e.g.: $N_0 = 16$

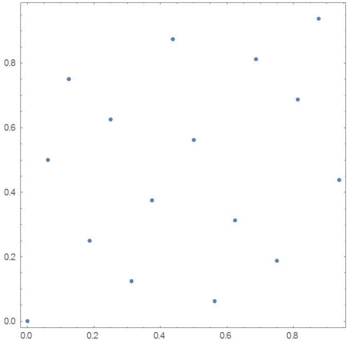


best configuration found until now

$$D^*_{16} = 0.4396 \cdot \frac{\log 16}{16}$$

Remark on 16-point set:

first approach:
best (0,4,2)-net in base 2



$$D^*_{16} = 0.9918 \cdot \frac{\log 16}{16}$$

$\Rightarrow$ "optimal" perturbation of net:

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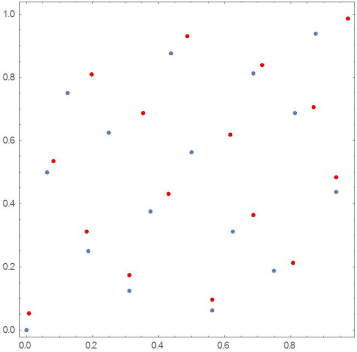

$$D_{16}^{*} = 0.51845 \cdot \frac{\log 16}{16}$$

(compare with $D_{16}^{*} = 0.4396 \cdot \frac{\log 16}{16}$)

$0.1203\ldots < c_\infty$
Bejian, 1982

$0.1211\ldots < c_\infty$
L., 2017

idea of proof:

K subinterval of $[0,1)$, $\quad U \in \mathbb{R}^+$

$$E_K(U) := \#\{1 \leq n \leq U \mid x_n \in K\} - U \cdot \lambda(K)$$

$$D_K(U) := \sup_{K' \subset K} |E_{K'}(U)|$$

$$M_K(N) := \frac{1}{N} \int_0^N D_K(U)\, dU$$

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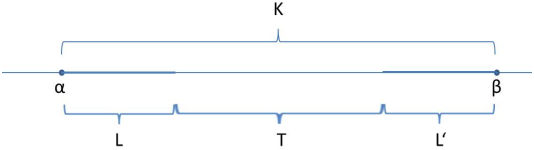
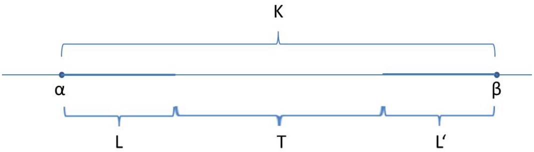
$length(L) = \frac{1}{a} \cdot length(K)$



$$M_K(N) \ge \frac{1}{2}\left(M_L(N) + M_{L'}(N)\right) + \frac{a-2}{4(a-1)}$$

main result!



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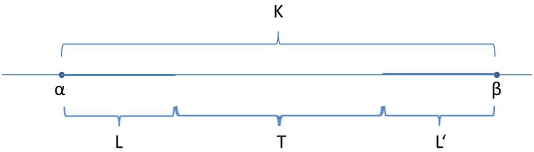

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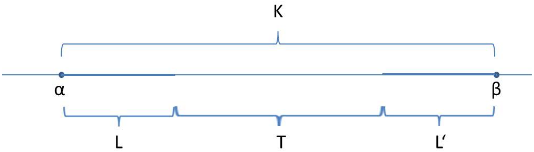

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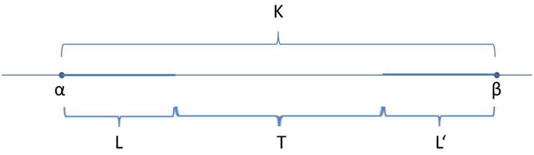
$\Rightarrow \exists\, K', U$

$$\frac{1}{N}\left|E_{K'}(U)\right| \geq \frac{\log N}{N} \cdot \frac{a-2}{4(a-1) \cdot \log a}$$



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optimal choice of a

$\Rightarrow$ result of Bejian

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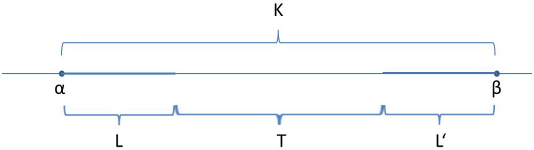

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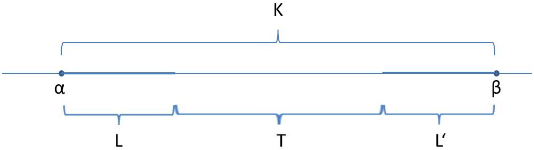
main result!

Cannot be improved! Best possible!

How to improve?



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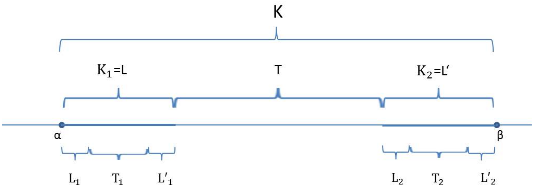
main result!

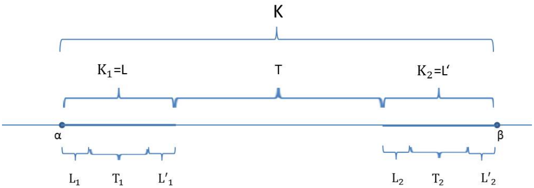
Cannot be improved! Best possible!

But: Cannot be best possible in two successive steps



Bejian's inequality $\Rightarrow$

$$M_K(N) \geq \frac{1}{4}\left(M_{L_1}(N) + M_{L'_1}(N) + M_{L_2}(N) + M_{L'_2}(N)\right) + $$
$$+ \quad 2 \cdot \frac{a-2}{4\cdot(a-1)}$$



Bejian's inequality $\Rightarrow$

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$$+ \quad 2 \cdot \frac{a-2}{4 \cdot (a-1)}$$

can be improved to

$$M_K(N) \geq \frac{1}{4}\left(M_{L_1}(N) + M_{L_1'}(N) + M_{L_2}(N) + M_{L_2'}(N)\right) + $$
$$+ \quad 2 \cdot \frac{a-2}{4 \cdot (a-1)} + \frac{1}{16a(a-1)^2(a+1)}$$

$\Rightarrow$ improvement of c_∞

Dear Henri!

Many Thanks for a lot of great results, and all the best for you and your family!