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Discrepancy Bounds for Nets and Sequences

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Introduction

Star discrepancy: measure for uniformity of distribution.

Let $\mathcal{P} := (\boldsymbol{x}_n)_{n=0}^{N-1}$ be a point set in $[0,1)^s$.

Star discrepancy of $\mathcal{P}$ is

$$D_N^*(\mathcal{P}) := \sup \left| \frac{|\{n : 0 \le n < N, \boldsymbol{x}_n \in [\boldsymbol{0}, \boldsymbol{a})\}|}{N} - \lambda_s([\boldsymbol{0}, \boldsymbol{a})) \right|,$$

where the supremum is extended over all $\boldsymbol{a} \in [0,1]^s$.

For an infinite sequence $\mathcal{S}$, $D_N^*(\mathcal{S})$ is the star discrepancy of the first N points of $\mathcal{S}$.

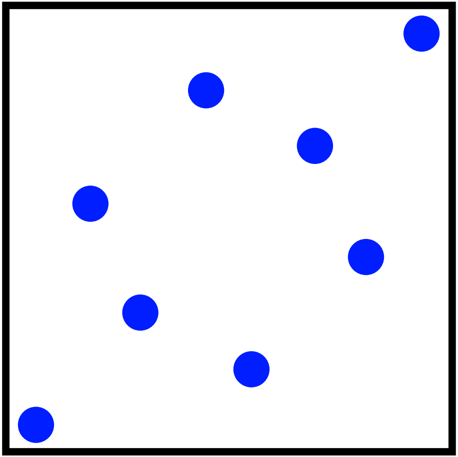
(t, m, s)-nets are evenly distributed point sets:

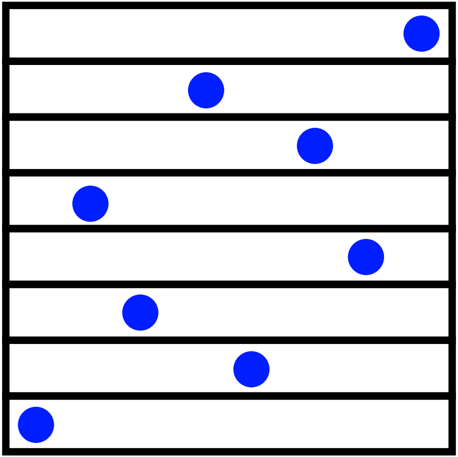
Let $b \geq 2$, $s \geq 1$, and $0 \leq t \leq m$ be integers. A point set $\mathcal{P}$ with $N = b^m$ points in $[0,1)^s$ is a (t, m, s)-net in base b, if every subinterval

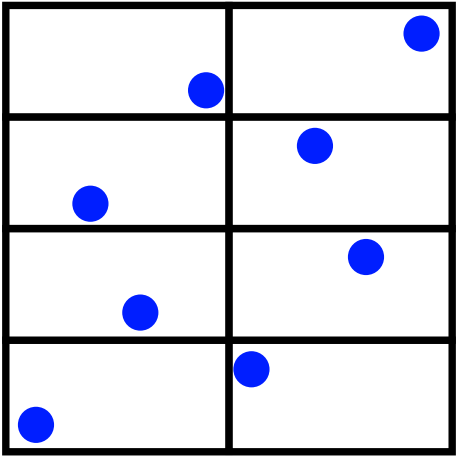
$$J = \prod_{j=1}^{s} \left[\frac{a_j}{b^{d_j}}, \frac{a_j + 1}{b^{d_j}} \right)$$

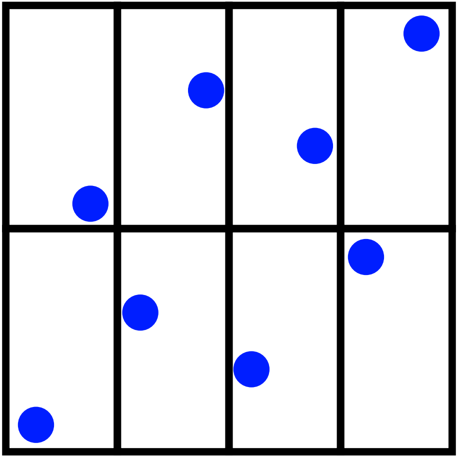
of $[0,1)^s$, with integers $d_j \geq 0$ and $0 \leq a_j < b^{d_j}$ for $1 \leq j \leq s$ and of volume b^{t-m}, contains exactly b^t points of $\mathcal{P}$.

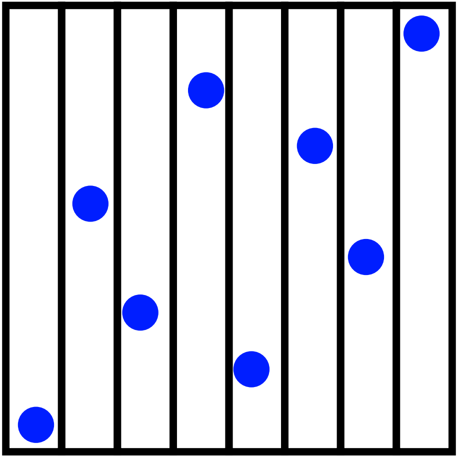
(based on concepts of Sobol', Faure, Niederreiter)











Infinite versions of (t, m, s)-nets: (t, s)-sequences:

Let $b \geq 2$, $s \geq 1$ and $t \geq 0$ be integers. An infinite sequence $\mathcal{S}$ is a (t, s)-sequence in base b, if for all $m > t$ and all $k \geq 0$ the points

$$\boldsymbol{x}_{kb^m}, \boldsymbol{x}_{kb^m+1}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}$$

form a (t, m, s)-net in base b.

Star discrepancy of (t,m,s)-nets and (t,s)-sequences?

- Lower bounds: few results (only for very special cases) apart from general bounds of Roth and Bilyk et al.
- Upper bounds: Much more is known.

Previous results

Numerous results by Faure for low-dimensional cases:

- In particular, results on $(0, 1)$-sequences, (generalized) van der Corput sequences, “NUT”-sequences, etc.
- Not only results on star discrepancy, but also L_p-discrepancies, diaphony.
- Results are based on a (very) detailed analysis of the local discrepancy function.

Further results by several other authors (e.g., Béjian, Chaix, DeClerck, Dick, Halton, Kritzer, Kritzinger, Larcher, Pausinger, Pillichshammer, Schmid, Zaremba, . . .).

Theorem 1 (Niederreiter, 1987)

Let $\mathcal{P}$ be a (t, m, s)-net in base b with $m > 0$, then

$$ND_N^*(\mathcal{P}) \leq B_{s,b} b^t (\log N)^{s-1} + \mathcal{O}\left(b^t (\log N)^{s-2}\right),$$

where the constant in the $\mathcal{O}$-notation does not depend on $N = b^m$, and where

$$B_{s,b} = \left(\frac{b-1}{2\log b}\right)^{s-1}$$

if $s = 2$, or $b = 2, s = 3, 4$; otherwise

$$B_{s,b} = \frac{1}{(s-1)!}\left(\frac{\lfloor b/2 \rfloor}{\log b}\right)^{s-1}.$$

Theorem 2 (Niederreiter, 1987)

Let $\mathcal{S}$ be a (t, s)-sequence in base b with $N \geq 2$, then

$$ND_N^*(\mathcal{S}) \leq C_{s,b} b^t (\log N)^s + \mathcal{O}\left(b^t (\log N)^{s-1}\right),$$

where the constant in the $\mathcal{O}$-notation does not depend on N, and where

$$C_{s,b} = \frac{1}{s}\left(\frac{b-1}{2\log b}\right)^s$$

if either $s = 2$ or $b = 2, s = 3, 4$; otherwise

$$C_{s,b} = \frac{1}{s!}\frac{b-1}{2\lfloor b/2\rfloor}\left(\frac{\lfloor b/2\rfloor}{\log b}\right)^s.$$

Method of proof: Split up the first N terms of $\mathcal{S}$ into nets and apply discrepancy bound for nets.

General bounds were sharpened with respect to $B_{s,b}$ and $C_{s,b}$:

- K., 2006: slight improvement using proof method as Niederreiter, but conjectured even better bounds.
- Faure & Lemieux, 2012: further improvement using an adaption of Atanassov's method.
- Faure & K., 2013: further refinement; new method for showing discrepancy bounds for (t, m, s)-nets. Conjecture of 2006 is true.

Results from paper with Henri, 2013:

Theorem 3 (Faure, K., 2013)

Let $s \geq 2$, $m \geq t \geq 0$, $b \geq 2$. Let $\mathcal{P}$ be a (t, m, s)-net in base b with $N = b^m$ points. Then it is true that

$$ND_N^*(\mathcal{P}) \leq \Delta_b(t, m, s),$$

where

$$\Delta_b(t, m, s) = b^t \sum_{v=0}^{s-1} a_{v,b}^{(s)} m^v,$$

where $a_{v,b}^{(s)}$ can be given explicitly.

Corollary 1 (Faure, K., 2013)

Let $\mathcal{P}$ be a (t, m, s)-net in base b with $N = b^m$ points. Then it is true that

$$ND_N^*(\mathcal{P}) \leq E_{s,b} b^t (\log N)^{s-1} + \mathcal{O}(b^t (\log N)^{s-2}) \text{ with}$$

$$E_{s,b} = \begin{cases} \frac{1}{(s-1)!} \frac{b^2}{2(b^2-1)} \left(\frac{b-1}{2\log b}\right)^{s-1} & \text{if } b \text{ is even,} \\ \frac{1}{(s-1)!} \frac{1}{2} \left(\frac{b-1}{2\log b}\right)^{s-1} & \text{if } b \text{ is odd.} \end{cases}$$

(Currently best known general upper bounds on discrepancy of (t, m, s)-nets with respect to leading coefficient $E_{s,b}$.)

Sketch of the proof of the theorem:

- Inspired by Niederreiter's proof method for (t, s)-*sequences*, modified to work also for (t, m, s)-*nets*.
- Based on induction on s, case $s = 2$ covered by an earlier result of Dick and K. (2006).
- Difference to proofs of results from 1987 (Niederreiter): no double induction on m and s.

In greater detail...

(1) Let $\mathcal{P}$ be an arbitrary (t,m,s)-net in base b with $N = b^m$ points $\boldsymbol{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$.

(2) Reorder and slightly move the points of $\mathcal{P}$ such that $x_n^{(1)} = n/N$ for $n \in \{0, \ldots, N-1\}$.

(3) Step (2) has negligible influence on the discrepancy of $\mathcal{P}$.

(4) For $1 \leq M \leq N$, let $\mathcal{R}_M$ be the first M points of the projection of $\mathcal{P}$ onto its last $s-1$ components.

(5) Now use

$$ND_N^*(\mathcal{P}) \leq \max_{1 \leq M \leq N} MD_M^*(\mathcal{R}_M) + 1.$$

(6) It remains to estimate $MD^*_M(\mathcal{R}_M)$.

(6a) Deal with special cases ($t = m$, $M = N$, $M < b^t$).

(6b) For the case $b^t \leq M < N$, consider the base b expansion,

$$M = \beta_0 + \beta_1 b + \cdots + \beta_k b^k.$$

(6c) Split up the points of $\mathcal{R}_M$ into subsequences $\omega_{\mu,\beta}$ for $\beta = 0, \ldots, \beta_\mu - 1$ and $\mu = 0, \ldots, k$, where $\omega_{\mu,\beta}$ contains those r_n of $\mathcal{R}_M$ with

$$\sum_{l=\mu+1}^{k} \beta_l b^l + \beta b^\mu \leq n < \sum_{l=\mu+1}^{k} \beta_l b^l + (\beta + 1) b^\mu.$$

(6d) Since we assumed for the first component of $\mathcal{P}$ that $x_n^{(1)} = n/N$ for $n \in \{0, \ldots, N-1\}$, it can be shown that the $\omega_{\mu,\beta}$ are $(t, \mu, s-1)$-nets in base b for $\mu \geq t$.

(6e) Using the “triangle inequality” for the discrepancy and induction on s yields the result.

Theorem 4 (Faure, K., 2013)

Let $s \geq 2$, $t \geq 0$, *let* $b \geq 2$, *and let* $\mathcal{S}$ *be a* (t, s)*-sequence in base* b*. Then*

$$ND_N^*(\mathcal{S}) \leq b^t \sum_{v=0}^{s} A_{v,b}^{(s)} (\log_b N)^v,$$

for any $N \geq \max\{b, b^t\}$, *where the coefficients* $A_{v,b}^{(s)}$ *can be given explicitly, in terms of the coefficients* $a_{v,b}^{(s)}$ *from the discrepancy bound for nets.*

Idea of the proof: Split up the first N terms of $\mathcal{S}$ into (t, m, s)-nets and apply the bound of nets (Niederreiter's method).

Corollary 2 (Faure, K., 2013)

Let $\mathcal{S}$ be a (t,s)-sequence in base b. Then,

$$ND_N^*(\mathcal{S}) \le D_{s,b} b^t (\log N)^s + \mathcal{O}(b^t (\log N)^{s-1}) \text{ with}$$

$$D_{s,b} = \begin{cases} \frac{1}{s!} \frac{b^2}{2(b^2-1)} \left(\frac{b-1}{2\log b}\right)^s & \text{if } b \text{ is even,} \\ \frac{1}{s!} \frac{1}{2} \left(\frac{b-1}{2\log b}\right)^s & \text{if } b \text{ is odd.} \end{cases}$$

(Currently best known general upper bounds on discrepancy of (t,s)-sequences with respect to leading coefficient $D_{s,b}$.)

Lower bounds

As stated before:

- Lower bounds much harder to obtain.
- But...
- Lower bounds for low-dimensional nets and sequences by Faure and his co-authors.
- Particular result from joint work with Henri, 2014: lower bound for special case of $(0, m, 2)$-nets.

Theorem 5 (Faure, K., 2014)

Let $\mathcal{P}$ *be a digital* $(0,m,2)$*-net over* $\mathbb{Z}_2$*, with generating matrices*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

where A *is an invertible* $\lfloor m/2 \rfloor \times \lfloor m/2 \rfloor$ *matrix.*
Then,

$$ND_N^*(\mathcal{P}) \geq m/12 + c,$$

where c *is a constant independent of* m*.*

Improvements?

- Tighter upper bounds for general (t, m, s)-nets and (t, s)-sequences?
- Lower bounds for general (t, m, s)-nets and (t, s)-sequences?

Project for the next 80 years.

Joyeux anniversaire, Henri!