

Tractability of Multivariate Approximation over Weighted Standard Sobolev Spaces

Henryk Woźniakowski

joint work with Arthur G. Werschulz, Fordham University,

Columbia University and University of Warsaw

Initiation

Thomas Kühn, Winfried Sickel and Tino Ullrich

Approximation numbers of Sobolev embeddings-
sharp constants and tractability

J. Complexity, 30, 95–116, 2014

Preliminaries

$$\mathcal{S} = \{S_d : \mathcal{H}_d \rightarrow \mathcal{G}_d\}_{d \in \mathbb{N}}.$$

Here, \mathcal{H}_d , \mathcal{G}_d Hilbert spaces, S_d compact linear

$$S_d(f) \approx A_{d,n}(f) = \phi_{d,n}(L_1(f), L_2(f), \dots, L_n(f))$$

with $\phi_n : \mathbb{C}^n \rightarrow \mathcal{G}_d$, $L_j \in \mathcal{H}_d^*$ can be adaptive.

The worst case setting:

$$e(A_{d,n}) = \sup_{\substack{f \in \mathcal{H}_d \\ \|f\|_{\mathcal{H}_d} \leq 1}} \|S_d(f) - A_{d,n}(f)\|_{\mathcal{G}_d}.$$

The n th minimal error

$$e(n, S_d) = \inf_{A_{d,n}} e(A_{d,n}).$$

Information Complexity

Information complexity:

$$n(\varepsilon, S_d) = \min\{n : e(n, S_d) \leq \varepsilon \|S_d\|\}.$$

Known:

$$W_d = S_d^* S_d : \mathcal{H}_d \rightarrow \mathcal{H}_d.$$

$$W_d \eta_{d,j} = \lambda_{d,j} \eta_{d,j}, \quad \lambda_{d,1} \geq \lambda_{d,2}, \dots$$

Then

$$n(\varepsilon, S_d) = \min\{n : \lambda_{d,n+1} \leq \varepsilon^2 \lambda_{d,1}\}. \quad (1)$$

Tractability: how does $n(\varepsilon, S_d)$ depends on d and ε^{-1} ?

Weighted Sobolev Spaces

$$\mathcal{T}_d = [0, 2\pi]^d \quad d\text{-torus}$$

$$\mathcal{G}_d = L_2(\mathcal{T}_d)$$

$$\Gamma = \bigcup_{d \in \mathbb{N}} \{ \gamma_{d,m} \in (0, 1] : m \in \mathbb{N}_0^d, |m| \leq r \}, \gamma_{d,0} = 1$$

$$\mathcal{H}_d = H_{\Gamma}^r(\mathcal{T}^d) = \{ f \in L_2(\mathcal{T}^d) : D^m f \in L_2(\mathcal{T}^d), |m| \leq r \}$$

with the inner product

$$\langle f, g \rangle_{H_{\Gamma}^r(\mathcal{T}^d)} = \sum_{|m| \leq r} \gamma_{d,m}^{-1} \langle D^m f, D^m g \rangle_{L_2(\mathcal{T}^d)}$$

Approximation Problem

$$S_d = f \quad \text{for all } f \in \mathcal{H}_d$$

Eigenvalues

For $k = [k_1, k_2, \dots, k_d] \in \mathbb{Z}^d$

$$\lambda_{d,k} = \frac{1}{1 + \sum_{1 \leq |m| \leq r} \gamma_{d,m}^{-1} \prod_{j=1}^d k_j^{2m}}$$

NOTE: Without periodicity conditions,
the exact values of $\lambda_{d,k}$ are unknown

First Tractability Results

- **SPT = strong polynomial tractability:**
 $n(\varepsilon, S_d)$ bounded by a polynomial in ε^{-1} independently of d ,
- **PT = polynomial tractability:**
 $n(\varepsilon, S_d)$ bounded by a polynomial in ε^{-1} and d ,
- **QPT = quasi-polynomial tractability:**
 $n(\varepsilon, S_d)$ bounded by a power of $\varepsilon^{-(1+\ln d)}$ for all d

$$\text{SPT} \implies \text{PT} \implies \text{QPT}$$

First Tractability Results

Theorem

No matter how the weights are chosen,
QPT (and SPT and PT) does not hold.

Note: For other spaces,

we usually have even SPT for sufficiently fast decaying weights

(s, t) -WT

- (s, t) -WT = (s, t) -weak tractability for positive s and t

$$\lim_{\varepsilon^{-1} + d \rightarrow \infty} \frac{\ln \max(1, n(\varepsilon, S_d))}{\varepsilon^{-s} + d^t} = 0$$

Theorem (Siedlecki and Weimar [2015])

Let $rs > 2$ or $t > 1$. Then

no matter how the weights are chosen,
 (s, t) -WT holds.

$$rs \in (0, 2] \text{ and } t \in (0, 1]$$

Product Weights:

$$\gamma_{d,m} = \prod_{j=1}^d \gamma_j \quad \text{for all } m \in \mathbb{N}_0^d \text{ and } d \in \mathbb{N}$$

Theorem

- Let $rs = 2$ and $t = 1$: (s, t) -WT iff $\gamma_j = o(1)$
- Let $rs = 2$ and $t < 1$: (s, t) -WT iff $\gamma_j = o([\ln j]^{-1})$
- Let $rs < 2$ and $t \leq 1$: (s, t) -WT iff $\gamma_j = o(j^{-(2-rs)/(rs)})$

UWT

- UWT = uniform weak tractability, Siedlecki [2013]
iff (s, t) -WT holds for all positive s and t .

Theorem

UWT iff $\gamma_j = o(j^{-p})$ for all $p > 0$.