On p-frame potential of random point configurations on the sphere

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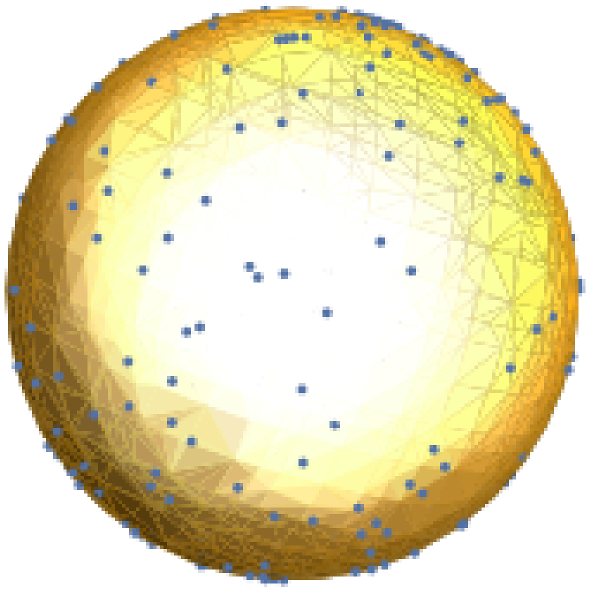

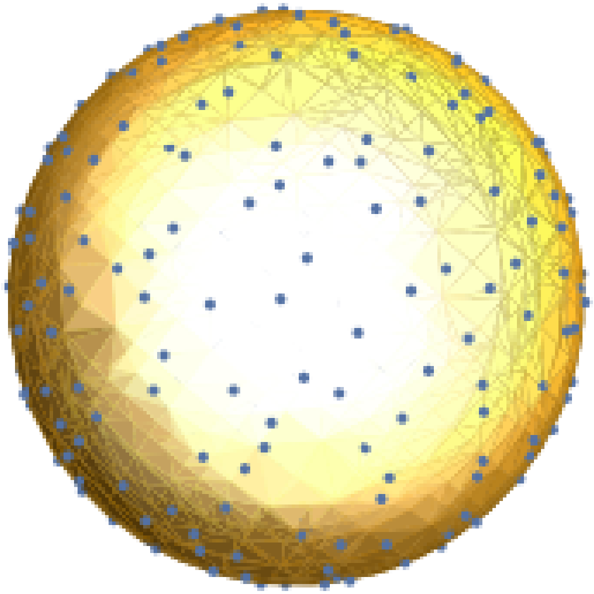

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My motivation

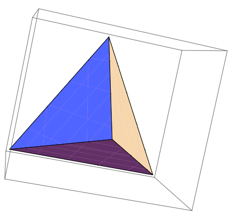
- We want to find "good" point config. like **spherical designs**.

$\mathbb{S}^d = \{\boldsymbol{x} \in \mathbb{R}^{d+1} \mid \|\boldsymbol{x}\| = 1\}$, $X_N = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \subset \mathbb{S}^d$, $N < \infty$

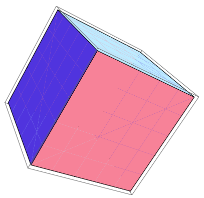
Def. A finite set X_N is called a spherical t-design of $\mathbb{S}^d$, if the following equation holds

$$\frac{1}{N}\sum_{i=1}^{N} f(\boldsymbol{x}_i) = \int_{\boldsymbol{x}\in\mathbb{S}^d} f(\boldsymbol{x})\, d\sigma(\boldsymbol{x}), \quad \forall f \in \mathcal{P}_t(\mathbb{R}^{d+1})$$

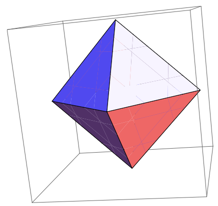
Ex. Vertices of a regular polyhedron form a spherical t-design of $\mathbb{S}^2$:



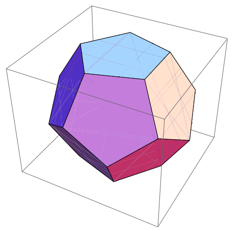
$t = 2$



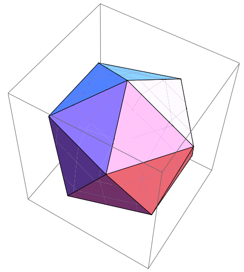
$t = 3$



$t = 3$



$t = 5$



$t = 5$

• There are many works on the existence of spherical design.

Thm. (Bondarenko et al., '13, '15). Given t, there always exists a spherical t-design X_N of $\mathbb{S}^d$ with $N = \mathcal{O}(t^d)$ $(t \to \infty)$.

• In general, it is **not easy** to construct such designs explicitly.

$\Rightarrow$ We focus on the following "good" random point config.

- ⋆ determinantal point processes(DPPs) $\leftarrow$ I mainly deal with this! Sorry!
- ⋆ jittered sampling

• We want to clarify **how different** these points and designs are!

Rem. These random config. form **QMC designs**. Roughly speaking, for sufficiently large N,

$$\frac{1}{N}\sum_{i=1}^{N} f(\boldsymbol{x}_i) \approx \int_{\boldsymbol{x}\in\mathbb{S}^d} f(\boldsymbol{x})\ d\sigma(\boldsymbol{x}), \quad \forall f \in \mathbb{H}^s(\mathbb{S}^d) \subset \mathbb{L}^2(\mathbb{S}^d)$$

⋆ These results were mentioned by J. Marzo (Title: Determinanal Point Processes and Optimality).

I will also touch about it if possible.

How to judge “good” point config.?

$\Rightarrow$ Today, we adopt p-frame potential as a measure of point config.

Def. (p-frame potential). $X_N = \{x_1, \dots, x_N\} \subset \mathbb{S}^d$, $0 < p < \infty$

$$\mathrm{FP}_p(X_N) = \sum_{i=1}^{N} \sum_{j=1}^{N} |\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle|^p$$

- There exist several works on the relationships between frame potential and spherical design.

Prop. (Seidelnikov's inequality).

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle^p \geq \begin{cases} N^2 \mathrm{PFP}(p), & p\text{: even} \\ 0, & p\text{: odd} \end{cases}$$

where $\mathrm{PFP}(p) = \int_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{S}^d} \langle \boldsymbol{x}, \boldsymbol{y} \rangle^p \, d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y})$, and

X_N is a spherical t-design $\Leftrightarrow$ “$=$” holds for any $p = 1, \dots, t$

DPP on the sphere (rough review)

- The concept of DPP is introduced by Macchi('74, '75).

- Let $\mathcal{X}$ be a simple point process on $\mathbb{S}^d$.
  - $*$ $\mathcal{X}$ can be identified with a **random discrete subset** of $\mathbb{S}^d$.

- Let $\mathcal{X}(D)$ be the number of points of this set that fall in $D \subset \mathbb{S}^d$.

$$\mathcal{X}(D) = \sum_i \delta_{\boldsymbol{x}_i}(D) \text{for any Borel set } D \subset \mathbb{S}^d$$

- Point processes are characterized by joint intensities.

Def. (joint intensities) For any $k \geq 1$, $\rho_k(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) : (\mathbb{S}^d)^k \to \mathbb{R}$, s.t., for any family of mutually disjoint sets $D_1, \ldots, D_k \subset \mathbb{S}^d$,

$$\mathbf{E}\left[\prod_i \mathcal{X}(D_i)\right] = \int_{D_1 \times \cdots \times D_k} \rho_k(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)\, d\sigma(\boldsymbol{x}_1) \cdots d\sigma(\boldsymbol{x}_k)$$

Def. DPP on $\mathbb{S}^d$ is a simple point process having deteminantal joint intensities

$$\rho_k(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \det \left[K(\boldsymbol{x}_i, \boldsymbol{x}_j) \right]_{i,j=1}^{k}, \quad \forall k \geq 1,\ \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{S}^d$$

for some function $K : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$

Ex. • **Harmonic ensembles on $\mathbb{S}^d$ with** $N = \dim(\mathcal{P}_L(\mathbb{S}^d))$ are induced by using the reproducing kernel $K(\boldsymbol{x}, \boldsymbol{y}) = Q_L(\langle \boldsymbol{x}, \boldsymbol{y} \rangle)$ of $\mathcal{P}_L(\mathbb{S}^d)$.

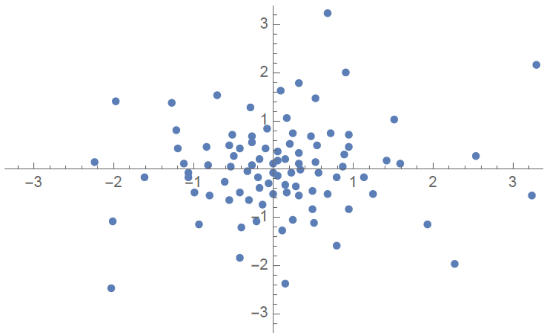
$$f(\boldsymbol{x}) = \int_{\boldsymbol{y} \in \mathbb{S}^d} Q_L(\langle \boldsymbol{x}, \boldsymbol{y} \rangle) f(\boldsymbol{y}) \, d\sigma(\boldsymbol{y}), \quad \forall f \in \mathcal{P}_L(\mathbb{S}^d)$$

$\star$ $Q_L(\cdot)$: Gegenbauer polynomial of degree L

• **spherical ensemble** (of $\mathbb{S}^2$):

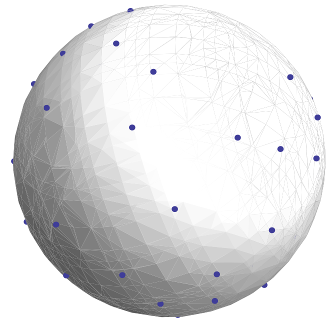
A_N, B_N: indep. $N \times N$ random matrices
with i.i.d. standard complex Gaussian entries

$\{\lambda_1, \ldots, \lambda_N\} \subset \mathbb{C}$: the set of eigenvalues of $A_N^{-1} B_N$



$\{\lambda_1, \ldots, \lambda_N\} \subset \mathbb{C}$

$\rightarrow$ $\boxed{\boldsymbol{x}_i = g^{-1}(\lambda_i)}$ $\rightarrow$



$\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathbb{S}^2$

g: the stereographic proj. from the north pole onto $\{(t_1, t_2, 0); t_1, t_2 \in \mathbb{R}\}$

$\star$ Set $g(\boldsymbol{x}) = z, g(\boldsymbol{y}) = w$, $\quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^2$, $\quad z, w \in \mathbb{C}$

$$K(\boldsymbol{x}, \boldsymbol{y}) = \frac{N(1 + z\bar{w})^{N-1}}{(1 + |z|)^{\frac{N-1}{2}}(1 + |w|)^{\frac{N-1}{2}}}$$

Key Lem. For any measurable $f : \mathbb{S}^2 \times \mathbb{S}^2 \to [0, \infty)$,

$$\mathbf{E}\left(\sum_{i \neq j} f(\boldsymbol{x}_i, \boldsymbol{x}_j)\right) = \int_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{S}^2} \rho_2(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{x}, \boldsymbol{y})\, d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y})$$

$$\rho_2(\boldsymbol{x}, \boldsymbol{y}) = N^2 - N^2 \Big(\frac{|\boldsymbol{x} - \boldsymbol{y}|^2}{4}\Big)^{N-1}, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^2 \quad \text{(spherical ensemble)}$$

$$\rho_2(\boldsymbol{x}, \boldsymbol{y}) = N^2 - Q_L(\langle \boldsymbol{x}, \boldsymbol{y} \rangle)^2, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^d \quad \text{(harmonic ensemble)}$$

- Thus, by using the above, we obtain

$$\mathbf{E}(\mathrm{FP}_p(X_N)) = \mathbf{E}\Big(\sum_{i=1}^{N} \sum_{j=1}^{N} |\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle|^p\Big) = N + \mathbf{E}\Big(\sum_{i \neq j} |\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle|^p\Big)$$

Known results

Recall Let p be an even. If X_N is a spherical t-design of $\mathbb{S}^d$,

$$\mathsf{FP}_p(X_N) = N^2 \mathsf{PFP}(p)$$

where $\mathsf{PFP}(p) = \int_{\mathbb{S}^d \times \mathbb{S}^d} \langle \boldsymbol{x}, \boldsymbol{y} \rangle^p \, d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y})$

Prop. If X_N stands for the N-point **poisson point process** on $\mathbb{S}^d$,

$$\mathbf{E}(\mathsf{FP}_p(X_N)) = N^2 \mathsf{PFP}(p) + N(1 - \mathsf{PFP}(p))$$

$$\left(= N^2 \mathsf{PFP}(p) + \mathcal{O}(N) \ (N \to \infty)\right)$$

⋆ $\mathbf{E}(\mathsf{FP}_2(X_N)) = \dfrac{N^2}{d+1} + \dfrac{dN}{d+1}$

⋆ Given p, $\mathbf{E}(\mathsf{FP}_p(X_N)) - N^2 \mathsf{PFP}(p) \to \infty \quad (N \to \infty)$

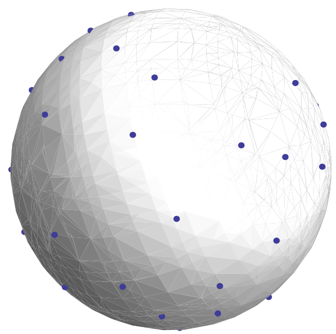
Main results: DPPs

Thm. (H.). If X_N stands for the N-**point spherical ensemble** on $\mathbb{S}^2$, then $\mathbf{E}(\mathrm{FP}_p(X_N)) = N^2\mathrm{PFP}(p)$

$$+N - \frac{N^2 B(N, p+1)}{2^N} - \frac{N^2}{2(p+1)} {}_2F_1(1, 1-N; p+2; \frac{1}{2})$$

$\star$ $\mathbf{E}(\mathrm{FP}_2(X_N)) = \frac{N^2}{3} + \frac{4N^2}{(N+1)(N+2)}$

$$\left(= \frac{N^2}{3} + \mathcal{O}(1) \quad (N \to \infty)\right)$$



$\star$ Given p,

$$\mathbf{E}(\mathrm{FP}_p(X_N)) - N^2\mathrm{PFP}(p) \to 2p \quad (N \to \infty)$$

Prob. Consider spherical ensembles on $\mathbb{S}^{2d}$, which introduced by Brauchart et al ('1*)!

Lem. If X_N stands for the N-**point harmonic ensemble** on $\mathbb{S}^d$ with $N = \dim(\mathcal{P}_L(\mathbb{S}^d)) = \binom{d+L}{d} + \binom{d+L-1}{d}$, then

$$\mathbf{E}(\mathsf{FP}_p(X_N)) = N^2 \mathsf{PFP}(p)$$

$$+N - \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|} \sum_{i,j=0}^{L} \int_{-1}^{1} |t|^p Q_i(t) Q_j(t) (1-t^2)^{\frac{d}{2}-1} dt \quad (*)$$

⋆ When p is a small integer, we can calculate $(*)$ explicitly.

Prop (H.) $\mathbf{E}(\mathsf{FP}_2(X_N))$

$$= \frac{N^2}{d+1} + \frac{d(-d+d^2+4dL+4L^2)}{(d+L)(d+2L+1)(d+2L-1)} \binom{d+L}{d}$$

$$\left(= \frac{N^2}{d+1} + O(N^{1-\frac{1}{d}}) \ (N \to \infty) \right)$$

Prob. Calculate $(*)$ for general p!

Main results: jittered sampling

Thm. (H.). Let $\{D_N\}$ be a seq. of partitions of $\mathbb{S}^d$ into N equal area subsets D_i $i = 1, \ldots, N$, s.t., $\mathrm{diam} D_i \leq c/N^{1/d}$.

Let $\underbrace{X_N = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}}_{\text{a set of points of jittered sampling}}$; $\boldsymbol{x}_i$ is chosen uniformly on D_i.

Then,

$$\mathbf{E}(\mathrm{FP}_p(X_N)) \leq N^2 \mathrm{PFP}(p) + N - N\left(1 - \frac{c^2}{2N^{2/d}}\right)^{p/2}$$

⋆ $\mathbf{E}(\mathrm{FP}_2(X_N)) \leq \dfrac{N^2}{d+1} + \dfrac{c^2}{2N^{2/d}}$

⋆ When $d = 2$,

$$\mathbf{E}(\mathrm{FP}_p(X_N)) - N^2 \mathrm{PFP}(p)$$

$$\leq N - N\left(1 - \frac{c^2}{2N}\right)^{p/2} \to \frac{pc^2}{4} \quad (N \to \infty)$$

Rem. We calculate p-frame potentials of DPPs (spherical & harmonic ensembles) and jittered sampling.

• frame potentials of DPPs, jittered sampling are smaller than PPP. Namely, if X_N stands for a DPP or jittered sampling on $\mathbb{S}^d$,

$$\mathbf{E}(\mathrm{FP}_p(X_N)) = N^2\mathrm{PFP}(p) + o(N) \quad (N \to \infty)$$

• spherical ensemble is a similar to jittered sampling on $\mathbb{S}^2$ in the sense of 2-frame potential. Namely, if X_N stands for Spherical ensemble or jittered sampling on $\mathbb{S}^2$,

$$\mathbf{E}(\mathrm{FP}_p(X_N)) = N^2\mathrm{PFP}(p) + \mathcal{O}(1) \quad (N \to \infty)$$

Application of 2-frame potential

$X_N = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathbb{S}^d$, $N < \infty$

Def. A finite set X_N is called a finite unit norm tight frame (FUNTF) for $\mathbb{R}^{d+1}$, if there is a positive constant A satisfying

$$\sum_{i=1}^{N} |\langle \boldsymbol{x}, \boldsymbol{x}_i \rangle|^2 = A\|\boldsymbol{x}\|^2, \quad \forall \boldsymbol{x} \in \mathbb{R}^{d+1}$$

$\ast$ A spherical 2-design is also a FUNTF.

$F : \mathbb{R}^{d+1} \to \mathbb{R}^N$, $\boldsymbol{x} \mapsto (\langle \boldsymbol{x}, \boldsymbol{x}_i \rangle)_{i=1}^N$: analysis op.

$F^* : \mathbb{R}^N \to \mathbb{R}^{d+1}$, $(c_i)_{i=1}^N \mapsto \sum_{i=1}^N c_i \boldsymbol{x}_i$: adjoint op. of F

Prop. (Cf. Christensen, '03).

X_N is a FUNTF for $\mathbb{R}^{d+1} \iff \frac{1}{N} F^* F = \frac{1}{d+1} \mathcal{I}_{d+1}$ ($\mathcal{I}_{d+1}$: identity mat.)

Prob. Find more applications of p-frames!

Thm. (Ehler, '12). Let $X_N = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathbb{S}^d$ be a set of random points, independently distributed according to $\underbrace{\text{prob. unit norm tight frames}\{\mu_k\}_{k=1}^N}$, respectively. Then

Ex. $\mu_k =$ the surface meas. σ on $\mathbb{S}^d$

$$\mathbf{E}(\|\frac{1}{N}F^*F - \frac{1}{d+1}\mathcal{I}_{d+1}\|_{\mathcal{F}}^2) = \frac{1}{N}\frac{d}{d+1}$$

where $\|\cdot\|_{\mathcal{F}}$ denotes the Frobenius norm.

Lem. $\mathbf{E}(\|\frac{1}{N}F^*F - \frac{1}{d+1}\mathcal{I}_{d+1}\|_{\mathcal{F}}^2) = \frac{1}{d+1} - \frac{1}{N^2}\mathbf{E}(\mathrm{FP}_2(X_N))$

Ex. If X_N is a set of N-PPP, recalling $\mathbf{E}[\mathrm{FP}_2(X_N)] = \frac{N^2}{d+1} + \frac{dN}{d+1}$, we obtain the same value.

Cor. (H.). (i) If X_N stands for the N-point spherical ensemble on $\mathbb{S}^2$, then

$$\mathbf{E}(\|\frac{1}{N}F^*F - \frac{1}{3}\mathcal{I}_3\|_{\mathcal{F}}^2) = \frac{4}{(N+1)(N+2)}$$

(ii) If X_N stands for the N-point harmonic ensemble on $\mathbb{S}^d$ with $N = \dim(\mathcal{P}_L(\mathbb{S}^d))$, then

$$\mathbf{E}(\|\frac{1}{N}F^*F - \frac{1}{d+1}\mathcal{I}_{d+1}\|_{\mathcal{F}}^2) = O(N^{-\frac{d+1}{d}}) \quad (N \to \infty)$$

(iii) If X_N stands for the N-point jittered sampling on $\mathbb{S}^d$, then

$$E(\|\frac{1}{N}F^*F - \frac{1}{d+1}\mathcal{I}_{d+1}\|_{\mathcal{F}}^2) \le \frac{c^2}{2N^{1+2/d}}$$

• The above of random point configurations give faster rates of convergence for the expected square Frobenius norm gap!

Conclusion

We calculate p-frame potentials of DPPs (spherical & harmonic ensembles) and jittered sampling.

• Frame potentials of DPPs, jittered sampling are smaller than PPP.

• Spherical ensemble is a similar to jittered sampling on $\mathbb{S}^2$ in the sense of 2-frame potential.

• DPPs, jittered sampling give faster rates of convergence for the expected square Frobenius norm gap.

Prob.

$\star$ Which is the best random point configurations on $\mathbb{S}^d$?

$\star$ How to simulate? $\star$ Efficient applications? etc

Thank you for your attension!

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