On the Dispersion of Sparse Grids

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Terminology

Point set:

$$P \subset [0,1]^d \quad \text{finite}$$

Boxes:

$$\mathcal{B}_d = \left\{ \prod_{i=1}^d J_i \mid J_i \subset [0,1] \;\; \text{interval} \right\}$$

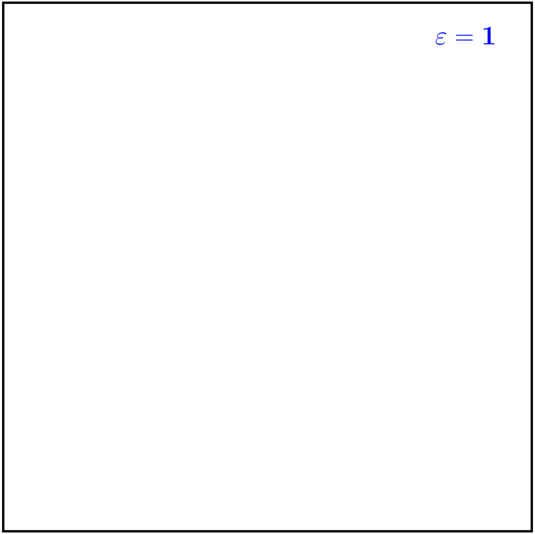
Dispersion:

$$\text{disp}(P) = \max \{ \text{vol}(B) \mid B \in \mathcal{B}_d : B \cap P = \emptyset \}$$

Task

Given $\varepsilon > 0$ and $d \in \mathbb{N}$.
Find $P \subset [0,1]^d$ with $\text{disp}(P) \leq \varepsilon$ such that the cardinality $|P|$ is as small as possible.

A natural candidate



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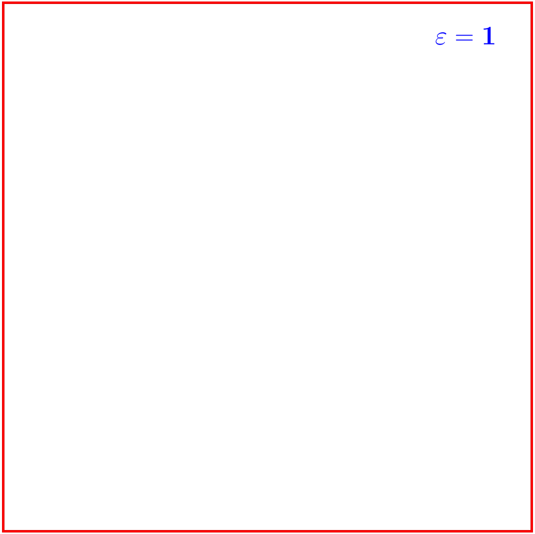

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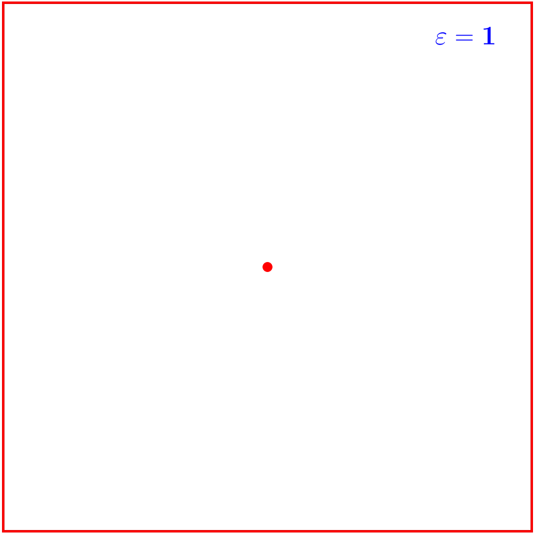

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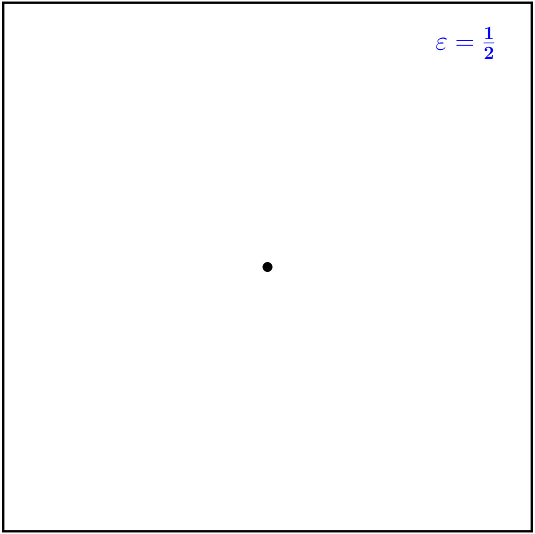

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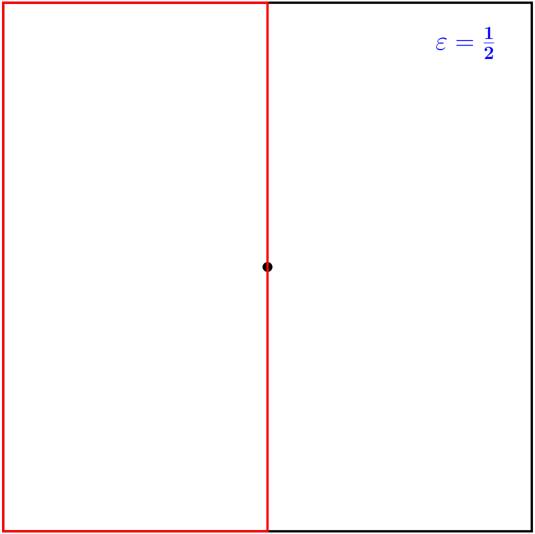

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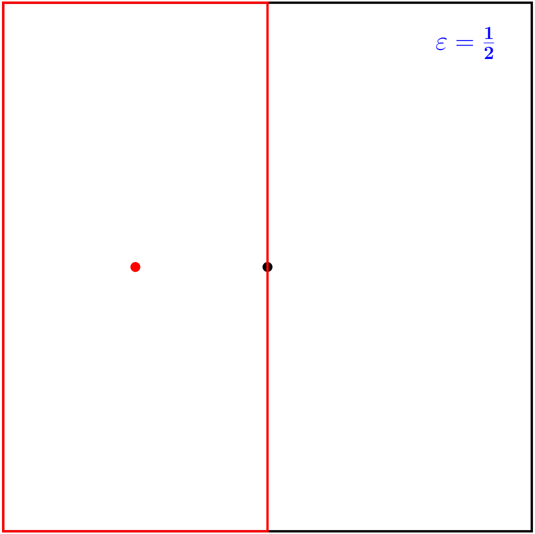

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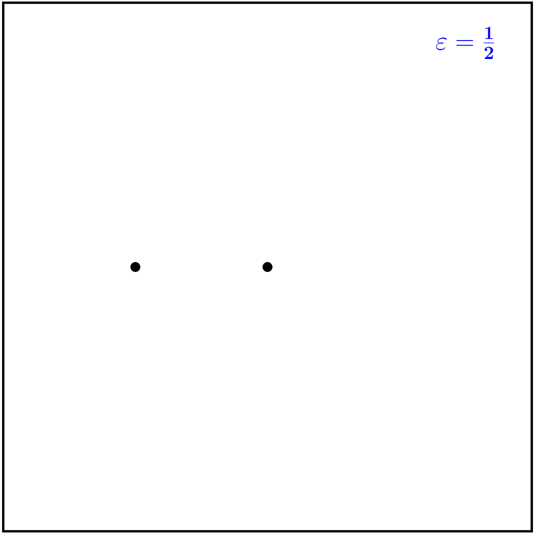

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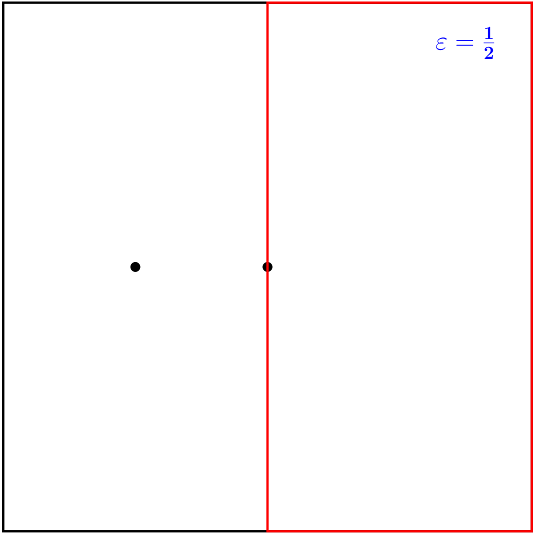

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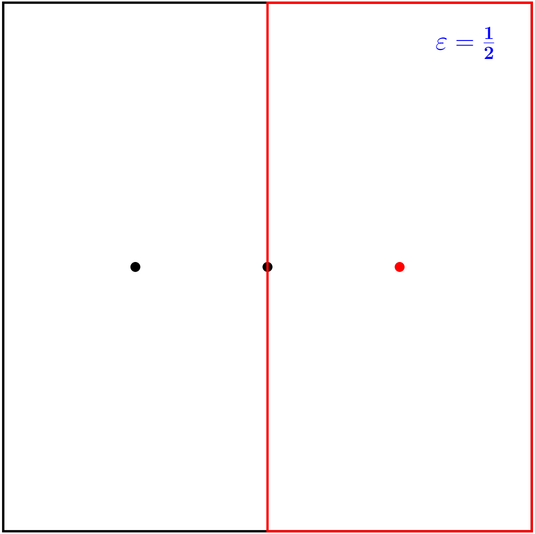

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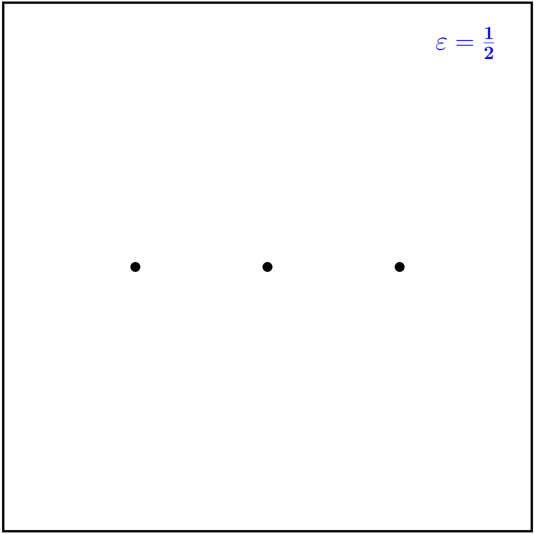

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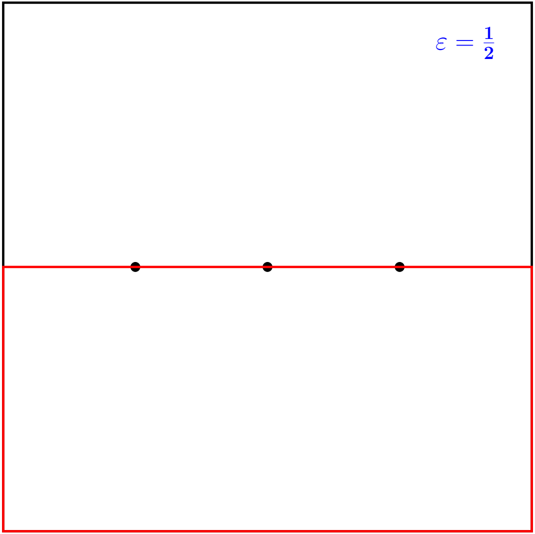

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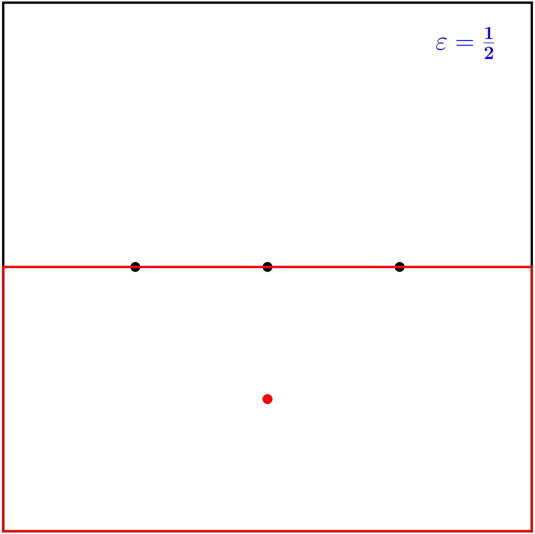

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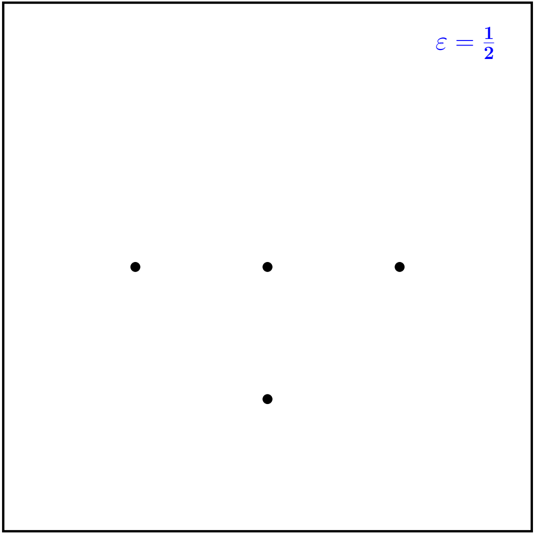

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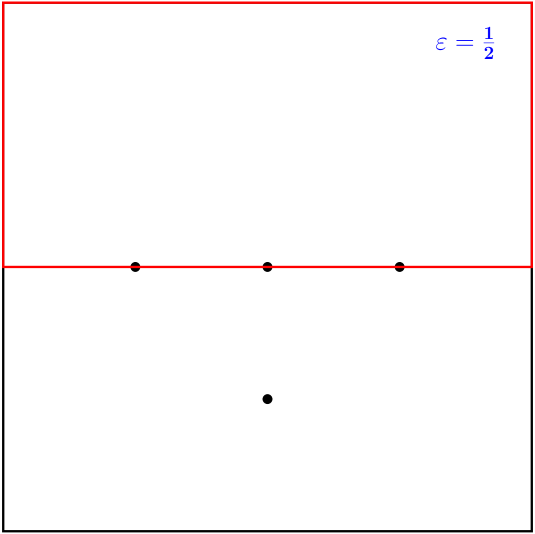

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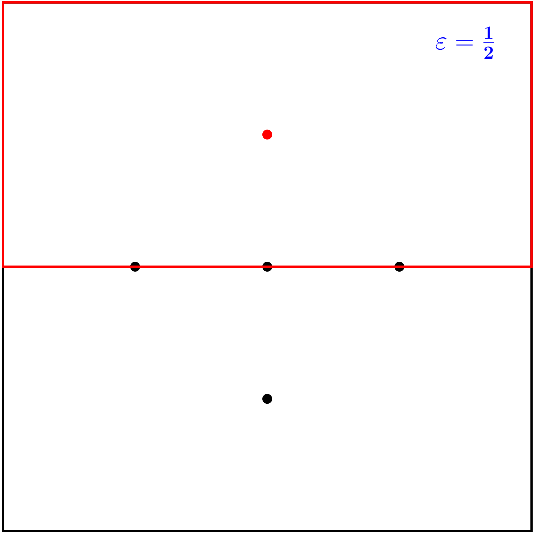

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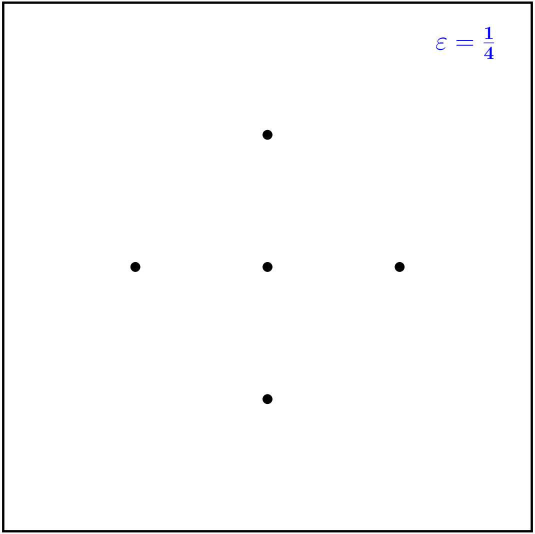

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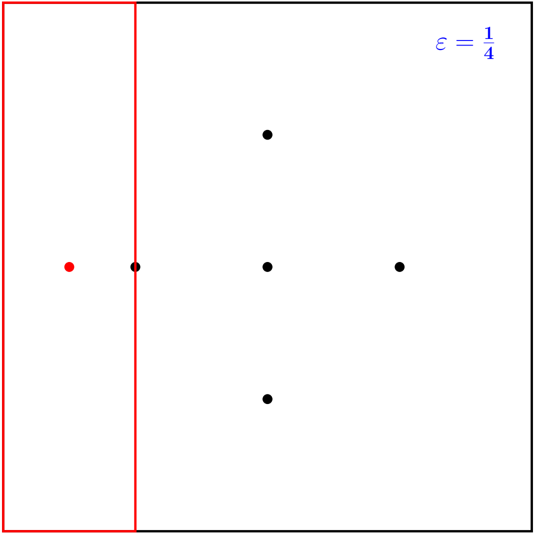

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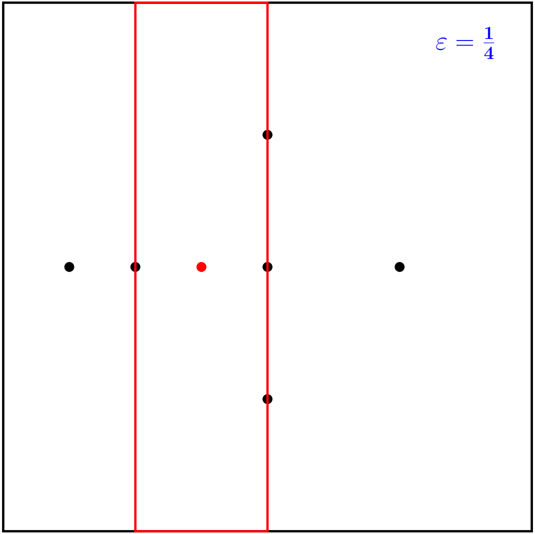

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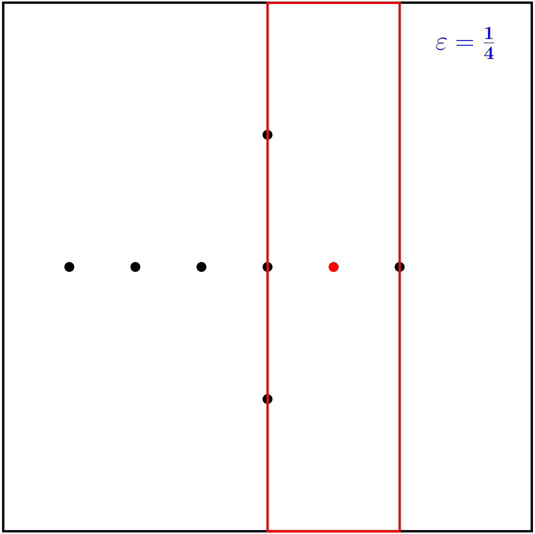

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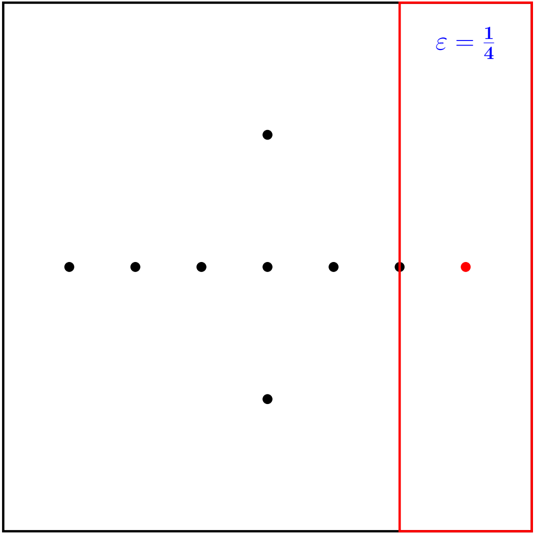

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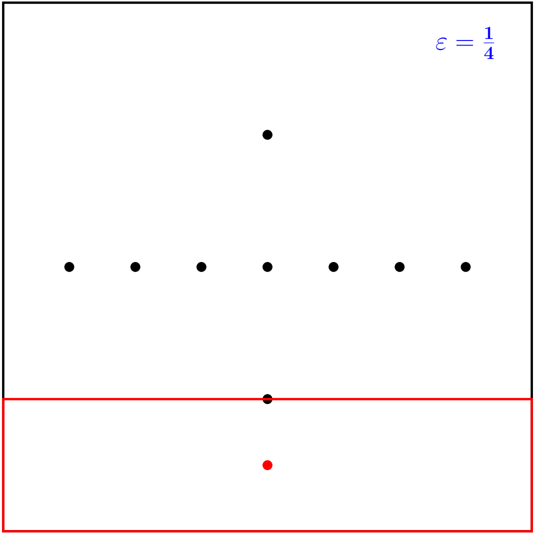

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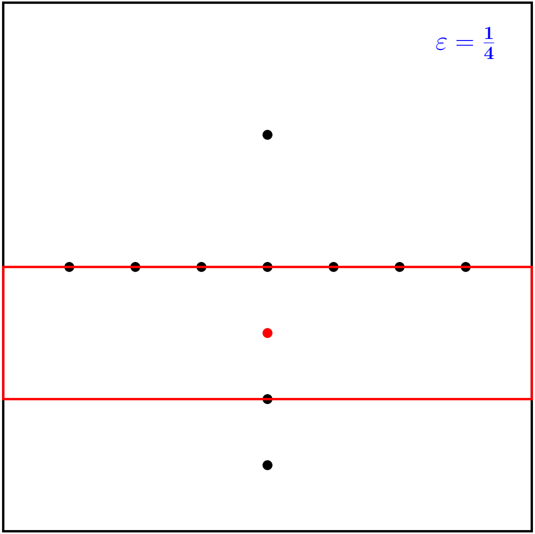

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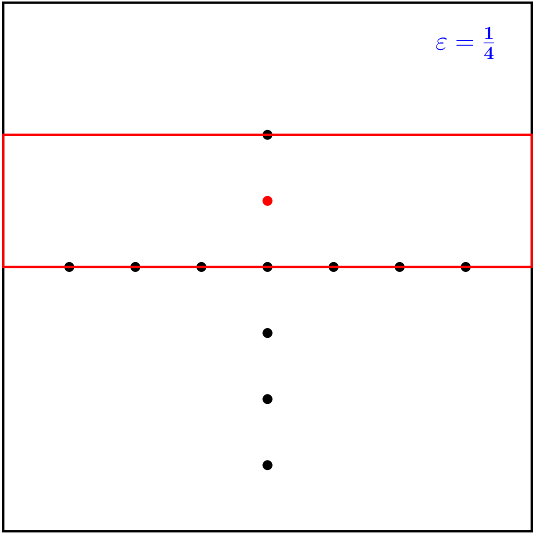

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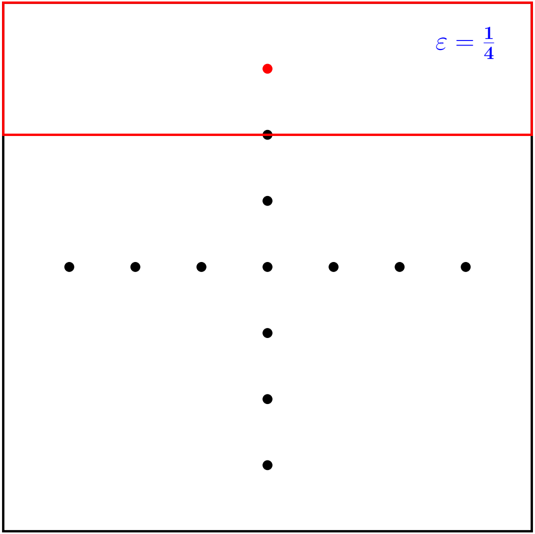

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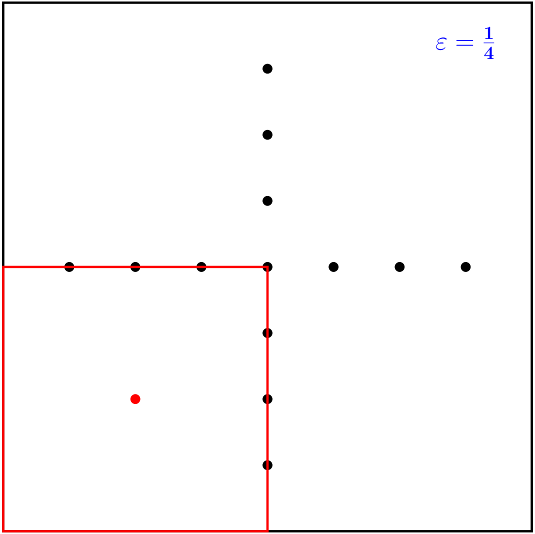

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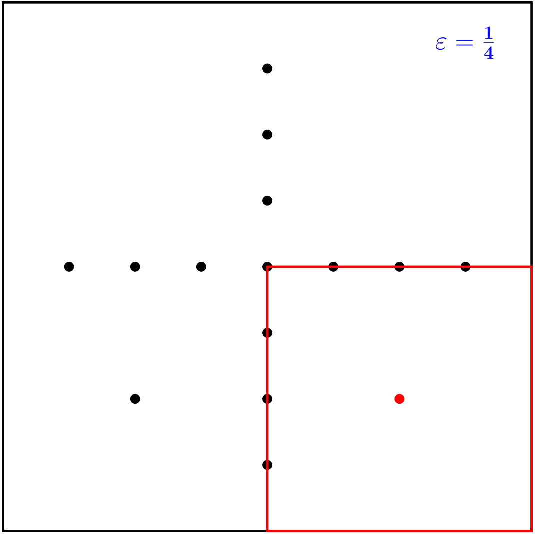

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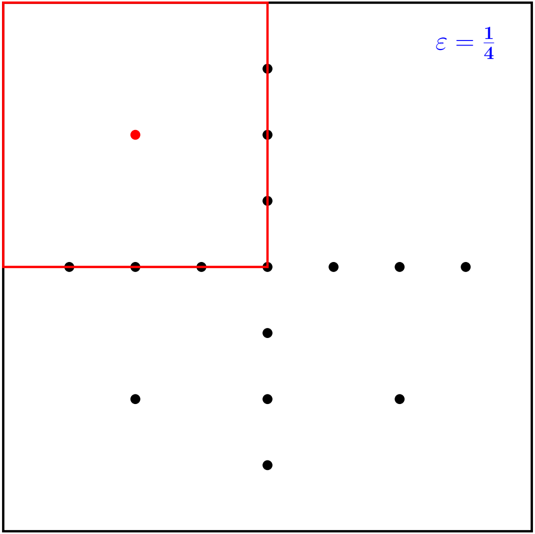

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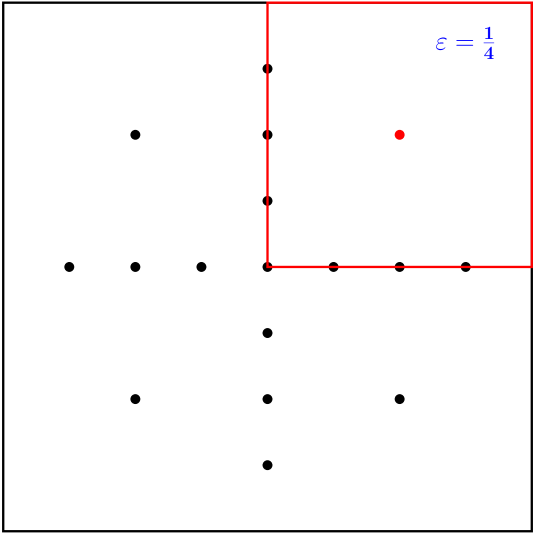

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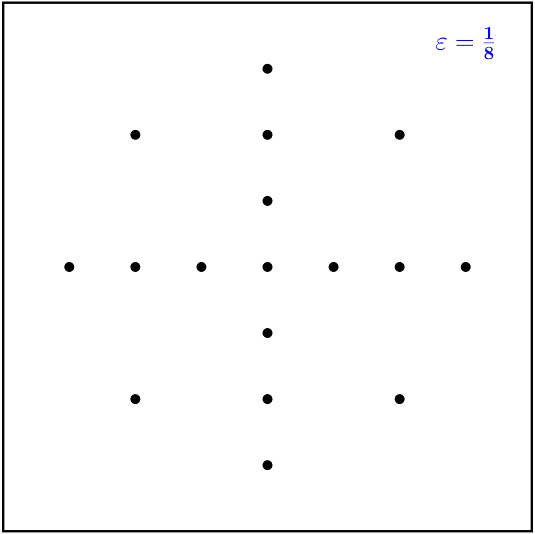

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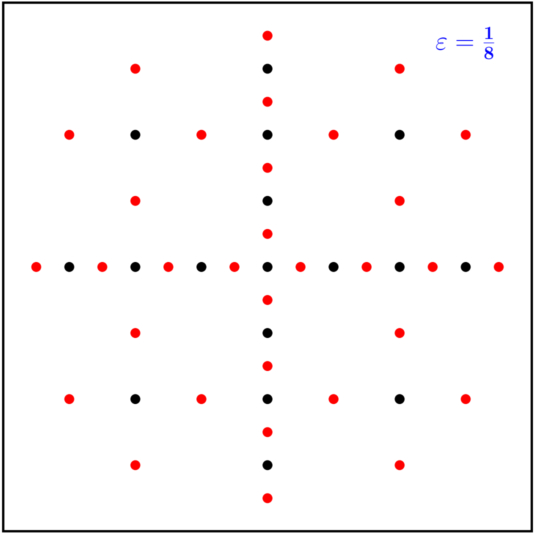
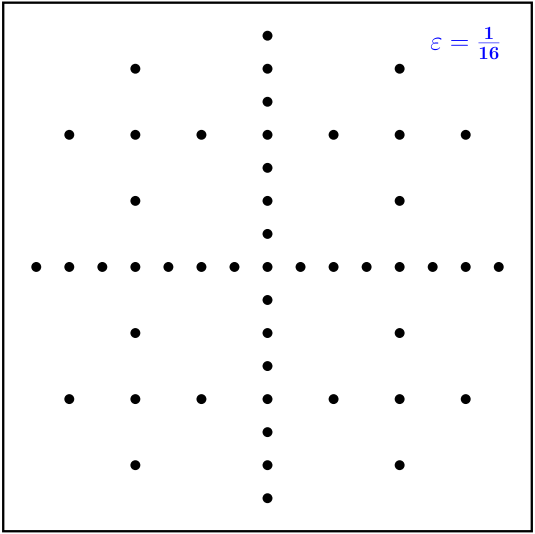


A natural candidate: The sparse grid



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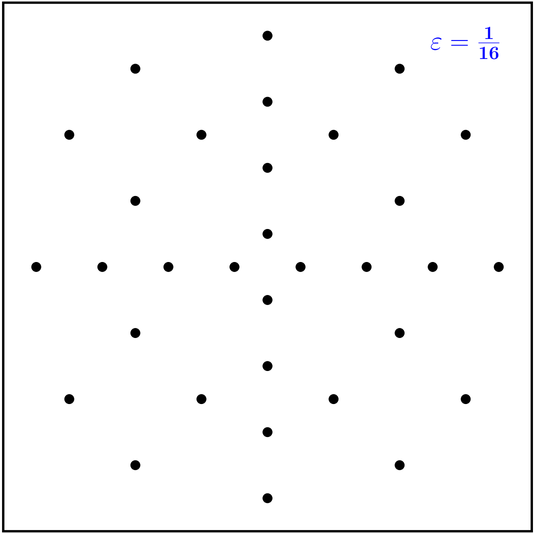


A natural candidate: The sparse grid

Definition. For $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^d$ let

$$\deg(t) = \min\left\{k \in \mathbb{N}_0 \mid 2^{k+1} t \in \mathbb{Z}\right\}$$

$$\deg(\boldsymbol{x}) = \sum_{\ell=1}^{d} \deg(x_\ell).$$

Then

$$P(k,d) = \left\{\boldsymbol{x} \in (0,1)^d \mid \deg(\boldsymbol{x}) = k\right\}.$$

Hence,

$$P(k,1) = \left\{\frac{1}{2^{k+1}}, \frac{3}{2^{k+1}}, \ldots, \frac{2^{k+1}-1}{2^{k+1}}\right\},$$

$$P(k,d) = \bigcup_{|\boldsymbol{i}|=k} \prod_{\ell=1}^{d} P(i_\ell, 1).$$

A good candidate?

Theorem

For all $k \in \mathbb{N}_0$ and $d \geq 2$, we have

$$a) \quad \mathrm{disp}(P(k,d)) = 2^{-(k+1)}, \quad \text{where}$$

$$b) \quad |P(k,d)| = 2^k \binom{d+k-1}{k}.$$

In particular:

For $\varepsilon > 0$ choose $k(\varepsilon) \in \mathbb{N}_0$ maximal with $2^{-k(\varepsilon)} > \varepsilon$.

Then $\mathrm{disp}(P(k(\varepsilon),d)) \leq \varepsilon$ and ...

A good candidate?

$$|P(k(\varepsilon), d)| \leq \varepsilon^{-1} \left\lceil \log_2 \left(\varepsilon^{-1}\right) \right\rceil^{d-1}$$

Optimal rate in ε^{-1}: $|P| \leq C_d \, \varepsilon^{-1}$.

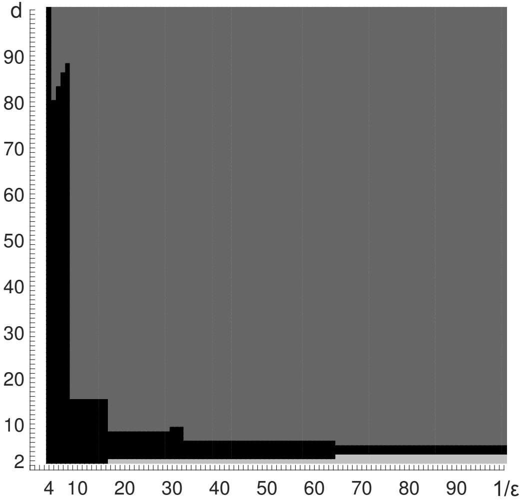
Achieved by: Halton-Hammersley set (Rote/Tichy '96),
(t,m,s)-nets (Larcher '17)

$$|P(k(\varepsilon), d)| \leq (2d)^{k(\varepsilon)}$$

Optimal rate in d: $|P| \leq C_\varepsilon \log_2(d)$.

Achieved by: ??? (Sosnovec '17, Ullrich/Vybíral '18)

But . . .



Dark & black:
Best known set.

Black:
Best known upper bound.

Challenge: For $d = 50$ and $\varepsilon = 1/60$, find a point set with

$$*)\ \operatorname{disp}(P) \leq \varepsilon, \qquad *)\ |P| < |P(k(\varepsilon), d)|.$$

Proof

Theorem

$$a)\quad \mathrm{disp}(P(k,d)) = 2^{-(k+1)}, \quad \text{where}$$
$$b)\quad |P(k,d)| = 2^k \binom{d+k-1}{k}.$$

$$|P(k,d)| = \sum_{|\boldsymbol{i}|=k} \prod_{\ell=1}^{d} |P(i_\ell, 1)| = \left| \left\{ \boldsymbol{i} \in \mathbb{N}_0^d \mid |\boldsymbol{i}| = k \right\} \right| \cdot 2^k.$$

Let $B = (0, 2^{-(k+1)}) \times (0,1)^{d-1}$. Then $B \cap P(k,d) = \emptyset$.

$$\hookrightarrow \quad \mathrm{disp}(P(k,d)) \geq \mathrm{vol}(B) = 2^{-(k+1)}.$$

Proof (Upper bound)

Let $B = J_1 \times \ldots \times J_d \in \mathcal{B}_d$ with $\mathrm{vol}(B) > 2^{-(k+1)}$.

For $\ell = 1 \ldots d$ choose $i_\ell \in \mathbb{N}_0$ with $2^{-(i_\ell+1)} < \mathrm{vol}(J_\ell) \le 2^{-i_\ell}$.

- $\mathrm{vol}(2^{i_\ell+1} J_\ell) > 1 \;\Rightarrow\;$ There is $x_\ell \in J_\ell$ with $\deg(x_\ell) \le i_\ell$, that is $\boldsymbol{x} = (x_1, \ldots, x_d) \in B$ and $\deg(\boldsymbol{x}) \le |\boldsymbol{i}|$.
- $\mathrm{vol}(B) \le 2^{-|\boldsymbol{i}|} \;\Rightarrow\; |\boldsymbol{i}| \le k$.

Hence: $\boldsymbol{x} \in B$, where $\deg(\boldsymbol{x}) = r \le k$.

Manipulate $\boldsymbol{x}$ such that $\deg(\boldsymbol{x}) = k$, if necessary.

- $\widetilde{x}_\ell := x_\ell \pm 2^{-(\deg(x_\ell)+1+k-r)} \in J_\ell$ for some $\ell \in \{1 \ldots d\}$.
  Else: $\mathrm{vol}(B) \le \prod_{\ell=1}^{d} 2^{-(\deg(x_\ell)+k-r)} = 2^{-k-(d-1)(k-r)}$ $\lightning$.
- $\deg(t) > \deg(s) \;\Rightarrow\; \deg(t+s) = \deg(t)$.
  Here: $\deg(\widetilde{x}_\ell) = \deg(x_\ell) + k - r \;\Rightarrow\; \deg(\widetilde{\boldsymbol{x}}) = k$.

Hence: $\widetilde{\boldsymbol{x}} \in B$, where $\deg(\widetilde{\boldsymbol{x}}) = k$.