Lower Error Bounds for Strong Approximation of Scalar SDEs with non-Lipschitzian Coefficients

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Joint work with
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Introduction

Given
A scalar SDE

\[ dX(t) = a(X(t)) \, dt + b(X(t)) \, dW(t), \quad t \in [0, 1], \]
\[ X(0) = \xi, \]

with coefficients \( a, b : \mathbb{R} \to \mathbb{R} \), one-dimensional driving Brownian motion \( W \), and initial value \( \xi \).
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Computational Problem

Approximate

\[ X : \Omega \to C([0, 1], \mathbb{R}) \quad \text{or} \quad X(1) : \Omega \to \mathbb{R} \]

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Goal
Establish lower error bounds in terms of \( n \) for non-Lipschitzian \( a, b \).
Literature on Lower Error Bounds for SDEs

(i) Coefficients are Lipschitz continuous

Literature on Lower Error Bounds for SDEs

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(ii) Coefficients are not Lipschitz continuous

- Subpolynomial lower error bounds
  

- CIR equation
  
The Class of Approximation Methods

$\mathcal{A}_n$ denotes the class of all approximations $\hat{X}: \Omega \rightarrow C([0, 1], \mathbb{R})$ of $X$ based on $n$ sequential point evaluations of $W$, on average.
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More Precisely
\( \hat{X} \in \mathcal{A}_n \) is given by measurable mappings \( \psi_k, \chi_k, \varphi_k, k \in \mathbb{N}, \) where

\[
\begin{align*}
\psi_k : \mathbb{R}^k &\to [0, 1] \quad (k\text{-th evaluation site}), \\
\chi_k : \mathbb{R}^{k+1} &\to \{ \text{Stop, Go} \} \quad (\text{stop evaluation of } W), \\
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Number of steps \( \nu = \min\{k \in \mathbb{N} : \chi_k(\xi, y_1, \ldots, y_k) = \text{Stop} \} \) with \( \mathbb{E}[\nu] \leq n \).
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Number of steps \( \nu = \min\{ k \in \mathbb{N} : \chi_k(\xi, y_1, \ldots, y_k) = \text{Stop} \} \) with \( \mathbb{E}[\nu] \leq n \).

Finally, \( \hat{X} = \varphi_\nu(\xi, y_1, \ldots, y_\nu) \).
Results

Theorem 1 (H., Herzwurm, Müller-Gronbach 2018)

Let $I \subset \mathbb{R}$ be open, $t_0 \in [0, 1)$, and assume

(i) $a, b$ are two times continuously differentiable on $I$,
(ii) for all $x \in I$: $b(x) \neq 0$,
(iii) $\mathbb{P}(X(t_0) \in I) > 0$.

Then there exists $c > 0$ such that for all $n \in \mathbb{N}$ and all $\hat{X}_n \in \mathcal{A}_n$

$$\mathbb{E}[\|X - \hat{X}_n\|_\infty] \geq c \, n^{-1/2} \cdot \sqrt{\ln(n)}$$
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Remark (Müller-Gronbach 2002)

The bounds are known and sharp (up to constants) if $a, b$ are two times cont. diff. on $\mathbb{R}$ with globally bounded derivatives.
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Let $I \subset \mathbb{R}$ be open, $t_0 \in [0, 1)$, and assume

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Remark (Yamato 1979)

If \( a, b \) are infinitely often differentiable and \( a'b - ab' - \frac{1}{2}b^2b'' = 0 \):

\[ X(1) = f(\xi, W(1)) \]

for some function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). Example: geometric BM.
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Applications: Polynomial Coefficients $a, b$

Assume
\[ \mathbb{P}(\xi \notin \text{zeros}(b) \cup \text{zeros}(a'b - ab' - \frac{1}{2}b^2b'')) > 0. \]
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Remark (Matching upper bounds for (i), (ii), (iii))
Under monotone conditions on $a, b$ and moment conditions on $\xi$ attained by non-sequential tamed/projected Euler, resp. Milstein schemes, see Hutzenthaler, Jentzen, Kloeden (2012, 2013), Gan, Wang (2013), Sabanis (2013, 2015), Beyn, Isaak, Kruse (2016a,b), Kumar, Sabanis (2016).
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Applications: CIR Equation

For $\beta \geq 0$, $x_0$, $\delta > 0$

$$dX(t) = (\delta - \beta \cdot X(t)) \, dt + 2\sqrt{|X(t)|} \, dW(t), \quad t \in [0, 1],$$

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Hence $a(x) = \delta - \beta \cdot x$ and $b(x) = 2\sqrt{|x|}$ for $x \in \mathbb{R}$. 
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On $(0, \infty)$, $a, b$ are infinitely often differentiable and $b$ is never zero. Furthermore, $\forall t \in [0, 1]: \mathbb{P}(X(t) \in (0, \infty)) = 1$. 
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Remark (Matching Upper Bounds)

For (i) and \( \delta \geq 2: \) Dereich, Neuenkirch, Szpruch (2011).
For (ii) and \( \delta > 1: \) H., Herzwurm (2017).
Applications: CIR Equation

For $x_0, \beta \geq 0, \delta > 0$

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**Remark**

$$(a' b - a b' - \frac{1}{2} b^2 b'')(x) = -\beta \sqrt{x} + (1 - \delta)/\sqrt{x} \text{ for } x \in (0, \infty).$$
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Case 1: $\beta \neq 0$ or $\delta \neq 1$

By Theorem 2: $\exists c > 0: \forall n \in \mathbb{N}, \hat{X}_n \in A_n$:

$$\mathbb{E}[|\hat{X}_n(1) - X(1)|] \geq c \, n^{-1}.$$ 

Matching upper bound for $\delta > 4$: Alfonsi (2013).
Applications: CIR Equation

For $x_0, \beta \geq 0, \delta > 0$

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Case 2: $\beta = 0$ and $\delta = 1$

Theorem 2 is not applicable.

Remark (H., Herzwurm 2017)

$\exists (\hat{X}_n \in A_n)_{n \in \mathbb{N}}: \forall r > 0 \exists c > 0:$

$$\mathbb{E}[|X(1) - \hat{X}_n(1)|] \leq cn^{-r}.$$
Applications: Discontinuous Drift Coefficient

\[ dX(t) = \text{sgn}(X(t))(1 + X(t))\,dt + dW(t), \quad t \in [0, 1], \]
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Hence \( a(x) = \text{sgn}(x)(1 + x) \) and \( b(x) = 1 \) for \( x \in \mathbb{R} \).
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**Remark**

On \( \mathbb{R} \setminus \{0\} \): \( a, b \) are infinitely often differentiable, \( b \neq 0 \), and \( a'b - ab' - \frac{1}{2} b^2 b'' = \text{sgn} \neq 0 \). \( \exists t \in [0, 1): \mathbb{P}(X(t) \neq 0) > 0. \)
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By Theorems 1, 2 there exists \( c > 0 \) s.t. \( \forall n \in \mathbb{N} \), \( \hat{X}_n \in A_n \), \( p \geq 1 \)

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Remark (Leobacher, Szölgyeny 2016)

\( \exists (\hat{X}_n \in A_n)_{n \in \mathbb{N}}, c > 0 : \)
\( \mathbb{E}[\|X - \hat{X}_n\|_{L^p}] \leq c \, n^{-1/2} \quad \text{and} \quad \mathbb{E}[|X(1) - \hat{X}_n(1)|] \leq c \, n^{-1/2}. \)
Idea of Proof

Step 1

Assume that $E[|\xi|^{16}] < \infty$ and $a, b$ are three times continuously differentiable with bounded derivatives and

$$\forall x \in \mathbb{R}: b(x) \neq 0 \text{ and } (a'b - ab' - \frac{1}{2}b^2b'')(x) \neq 0.$$ 

Then $\forall \varepsilon > 0 \exists c > 0: \forall n \in \mathbb{N}, \hat{X}_n \in \mathcal{A}_n$:

$$\mathbb{P}(|X(1) - \hat{X}_n(1)| \geq c n^{-1}) \geq 1 - \varepsilon.$$
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Step 2

Under the assumptions of Theorem 2: By standard arguments and a support theorem for diffusion processes, see Pakkanen (2010), $\exists c, \gamma > 0: \forall n \in \mathbb{N}, \hat{X}_n \in A_n$:

$$\mathbb{P}(|X(1) - \hat{X}_n(1)| \geq c n^{-1}) \geq \gamma.$$
Summary

\[ dX(t) = a(X(t)) \, dt + b(X(t)) \, dW(t), \quad t \in [0, 1], \]

\[ X(0) = \xi. \]
Summary

\[ dX(t) = a(X(t)) \, dt + b(X(t)) \, dW(t), \quad t \in [0, 1], \]
\[ X(0) = \xi. \]

**Theorem**

Let \( I \subset \mathbb{R} \) be open, \( t_0 \in [0, 1) \), and assume

(i) \( a, b \) are three times continuously differentiable on \( I \),
(ii) for all \( x \in I \): \( b(x) \neq 0 \) and \( (a'b - ab' - \frac{1}{2} b^2 b'')(x) \neq 0 \),
(iii) \( \mathbb{P}(X(t_0) \in I) > 0 \).

Then there exists \( c > 0 \) such that for all \( n \in \mathbb{N} \) and all \( \hat{X}_n \in \mathbb{A}_n \)

(i) \( \mathbb{E}[||\hat{X}_n - X||_{\infty}] \geq c \, n^{-1/2} \cdot \sqrt{\ln(n)}, \)
(ii) \( \mathbb{E}[||\hat{X}_n - X||_{L_p}] \geq c \, n^{-1/2}, \)
(iii) \( \mathbb{E}[|\hat{X}_n(1) - X(1)|] \geq c \, n^{-1}. \)
Summary
\[ dX(t) = a(X(t)) \, dt + b(X(t)) \, dW(t), \quad t \in [0, 1], \]
\[ X(0) = \xi. \]

Theorem
Let \( I \subset \mathbb{R} \) be open, \( t_0 \in [0, 1) \), and assume
\begin{enumerate}[(i)]
    
    \item \( a, b \) are three times continuously differentiable on \( I \),
    \item for all \( x \in I \): \( b(x) \neq 0 \) and \((a' b - ab' - \frac{1}{2} b^2 b'')(x) \neq 0\),
    \item \( \mathbb{P}(X(t_0) \in I) > 0 \).
\end{enumerate}

Then there exists \( c > 0 \) such that for all \( n \in \mathbb{N} \) and all \( \hat{X}_n \in \mathcal{A}_n \)
\begin{enumerate}[(i)]
    
    \item \( \mathbb{E}[\| \hat{X}_n - X \|_\infty] \geq c \, n^{-1/2} \cdot \sqrt{\ln(n)} \),
    \item \( \mathbb{E}[\| \hat{X}_n - X \|_{L_p}] \geq c \, n^{-1/2} \),
    \item \( \mathbb{E}[|\hat{X}_n(1) - X(1)|] \geq c \, n^{-1} \).
\end{enumerate}

Remark
Non-autonomous coefficients \( a, b : [0, 1] \times \mathbb{R} \to \mathbb{R} \) also covered.