Lower Error Bounds for Strong Approximation of Scalar SDEs with non-Lipschitzian Coefficients

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Joint work with
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Introduction

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A scalar SDE

$$\begin{aligned} \mathrm{d}X(t) &= a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}W(t), \quad t \in [0,1], \\ X(0) &= \xi, \end{aligned}$$

with coefficients $a, b\colon \mathbb{R} \to \mathbb{R}$, one-dimensional driving Brownian motion W, and initial value ξ.

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Approximate

$$X\colon \Omega \to C([0,1],\mathbb{R}) \quad \text{or} \quad X(1)\colon \Omega \to \mathbb{R}$$

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Goal

Establish lower error bounds in terms of n for non-Lipschitzian a, b.

Literature on Lower Error Bounds for SDEs

(i) Coefficients are Lipschitz continuous

- *Clark, Cameron (1980), Rümelin (1982), Cambanis, Hu (1996), Hofmann, Müller-Gronbach, Ritter (2000, ...), Müller-Gronbach (2002, ...), Neuenkirch (2006, ...)[frac.BM], Hofmann, Müller-Gronbach (2006)[sdde], Przybylowicz (2013, ...)[discont. in time]*

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(ii) Coefficients are not Lipschitz continuous
  - Subpolynomial lower error bounds
    Hairer, Hutzenthaler, Jentzen (2015), Jentzen, Müller-Gronbach, Yaroslavtseva (2016), Gerencsér, Jentzen, Salimova (2017), Müller-Gronbach, Yaroslavtseva (2017), Yaroslavtseva (2017)
  - CIR equation
    H., Herzwurm (2017), H., Jentzen (2017)

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More Precisely

$\widehat{X} \in \mathcal{A}_n$ is given by measurable mappings $\psi_k, \chi_k, \varphi_k$, $k \in \mathbb{N}$, where

$$\begin{aligned}
&\psi_k\colon \mathbb{R}^k \to [0,1] && (k\text{-th evaluation site}),\\
&\chi_k\colon \mathbb{R}^{k+1} \to \{\text{Stop}, \text{Go}\} && (\text{stop evaluation of } W),\\
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Evaluation site	Data
$\psi_1(\xi)$	$\xi, \quad y_1 = W(\psi_1(\xi))$
$\psi_k(\xi, y_1, \ldots, y_{k-1})$	$\xi, y_1, \ldots, y_{k-1}, \quad y_k = W(\psi_k(\xi, y_1, \ldots, y_{k-1}))$

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Number of steps $\nu = \min\{k \in \mathbb{N}\colon \chi_k(\xi, y_1, \dots, y_k) = \text{Stop}\}$ with $\mathbb{E}[\nu] \le n$.

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Number of steps $\nu = \min\{k \in \mathbb{N}\colon \chi_k(\xi, y_1, \ldots, y_k) = \text{Stop}\}$ with $\mathbb{E}[\nu] \le n$.
Finally, $\widehat{X} = \varphi_\nu(\xi, y_1, \ldots, y_\nu)$.

Results

Theorem 1 (H., Herzwurm, Müller-Gronbach 2018)

Let $I \subset \mathbb{R}$ *be open,* $t_0 \in [0,1)$*, and assume*

(i) a, b *are two times continuously differentiable on* I*,*

(ii) *for all* $x \in I$*:* $b(x) \neq 0$*,*

(iii) $\mathbb{P}(X(t_0) \in I) > 0$*.*

Then there exists $c > 0$ *such that for all* $n \in \mathbb{N}$ *and all* $\widehat{X}_n \in \mathcal{A}_n$

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Remark (Müller-Gronbach 2002)

The bounds are known and sharp (up to constants) if a, b are two times cont. diff. on $\mathbb{R}$ with globally bounded derivatives.

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Remark (Yamato 1979)

If a, b are infinitely often differentiable and $a'b - ab' - \frac{1}{2}b^2 b'' = 0$:

$$X(1) = f(\xi, W(1))$$

for some function $f\colon \mathbb{R}^2 \to \mathbb{R}$. Example: geometric BM.

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Applications: Polynomial Coefficients a, b

Assume

$\mathbb{P}(\xi \notin \text{zeros}(b) \cup \text{zeros}(a'b - ab' - \frac{1}{2}b^2 b'')) > 0.$

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By Theorems 1, 2: $\exists c > 0 \; \forall \, n \in \mathbb{N}, \widehat{X}_n \in \mathcal{A}_n, p \geq 1$:

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Remark (Matching upper bounds for (i), (ii), (iii))

Under monotone conditions on a, b and moment conditions on ξ attained by non-sequential tamed/projected Euler, resp. Milstein schemes, *see Hutzenthaler, Jentzen, Kloeden (2012, 2013), Gan, Wang (2013), Sabanis (2013, 2015), Beyn, Isaak, Kruse (2016a,b), Kumar, Sabanis (2016).* For an adaptive method, *see Fang, Giles (2016).*

Applications: CIR Equation

For $\beta \geq 0$, $x_0, \delta > 0$

$$\mathrm{d}X(t) = (\delta - \beta \cdot X(t))\,\mathrm{d}t + 2\sqrt{|X(t)|}\,\mathrm{d}W(t), \quad t \in [0,1],$$
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Remark (Matching Upper Bounds)

For (i) and $\delta \geq 2$: *Dereich, Neuenkirch, Szpruch (2011)*.
For (ii) and $\delta > 1$: *H., Herzwurm (2017)*.

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$(a'b - ab' - \frac{1}{2}b^2 b'')(x) = -\beta\sqrt{x} + (1-\delta)/\sqrt{x}$ for $x \in (0,\infty)$.

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Case 1: $\beta \neq 0$ or $\delta \neq 1$

By Theorem 2: $\exists\, c > 0 \colon \forall\, n \in \mathbb{N}, \widehat{X}_n \in \mathcal{A}_n$:

$$\mathbb{E}[|\widehat{X}_n(1) - X(1)|] \geq c\, n^{-1}.$$

Matching upper bound for $\delta > 4$: Alfonsi (2013).

Applications: CIR Equation

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Case 2: $\beta = 0$ and $\delta = 1$

Theorem 2 ist not applicable.

Remark (H., Herzwurm 2017)

$\exists\,(\widehat{X}_n \in \mathcal{A}_n)_{n\in\mathbb{N}}\colon \forall r > 0\,\exists\, c > 0\colon$

$$\mathbb{E}[|X(1) - \widehat{X}_n(1)|] \leq cn^{-r}.$$

Applications: Discontinuous Drift Coefficient

$$\mathrm{d}X(t) = \mathrm{sgn}(X(t))(1 + X(t))\,\mathrm{d}t + \mathrm{d}W(t), \quad t \in [0,1],$$
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On $\mathbb{R} \setminus \{0\}$: a, b are infinitely often differentiable, $b \neq 0$, and $a'b - ab' - \frac{1}{2}b^2 b'' = \mathrm{sgn} \neq 0$. $\exists t \in [0,1)\colon \mathbb{P}(X(t) \neq 0) > 0$.

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Remark (Leobacher, Szölgyeny 2016)

$\exists\, (\widehat{X}_n \in \mathcal{A}_n)_{n \in \mathbb{N}}, c > 0 \colon$
$\mathbb{E}[\|X - \widehat{X}_n\|_{L_p}] \leq c\, n^{-1/2}$ and $\mathbb{E}[|X(1) - \widehat{X}_n(1)|] \leq c\, n^{-1/2}$.

Idea of Proof

Step 1

Assume that $\mathbb{E}[|\xi|^{16}] < \infty$ and a, b are three times continuously differentiable with bounded derivatives and

$$\forall\, x \in \mathbb{R}: \qquad b(x) \neq 0 \quad \text{and} \quad (a'b - ab' - \tfrac{1}{2} b^2 b'')(x) \neq 0.$$

Then $\forall\, \varepsilon > 0\, \exists\, c > 0 \colon \forall\, n \in \mathbb{N}, \widehat{X}_n \in \mathcal{A}_n \colon$

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Step 2

Under the assumptions of Theorem 2: By standard arguments and a support theorem for diffusion processes, see Pakkanen (2010), $\exists\, c, \gamma > 0 \colon \forall\, n \in \mathbb{N}, \widehat{X}_n \in \mathcal{A}_n \colon$

$$\mathbb{P}\big(|X(1) - \widehat{X}_n(1)| \geq c\, n^{-1}\big) \geq \gamma.$$

Summary

$$\mathrm{d}X(t) = a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}W(t), \quad t \in [0,1],$$
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Summary

$$\mathrm{d}X(t) = a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}W(t), \quad t \in [0,1],$$
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Theorem

Let $I \subset \mathbb{R}$ be open, $t_0 \in [0,1)$, and assume

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(ii) for all $x \in I$: $b(x) \neq 0$ and $(a'b - ab' - \frac{1}{2}b^2 b'')(x) \neq 0$,

(iii) $\mathbb{P}(X(t_0) \in I) > 0$.

Then there exists $c > 0$ such that for all $n \in \mathbb{N}$ and all $\widehat{X}_n \in \mathcal{A}_n$

(i) $\mathbb{E}[\|\widehat{X}_n - X\|_\infty] \geq c\, n^{-1/2} \cdot \sqrt{\ln(n)}$,

(ii) $\mathbb{E}[\|\widehat{X}_n - X\|_{L_p}] \geq c\, n^{-1/2}$,

(iii) $\mathbb{E}[|\widehat{X}_n(1) - X(1)|] \geq c\, n^{-1}$.

Summary

$$\mathrm{d}X(t) = a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}W(t), \quad t \in [0,1],$$
$$X(0) = \xi.$$

Theorem

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Remark

Non-autonomous coefficients $a, b\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ also covered.